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SONAR TARGET MOTION ANALYSIS; WEIGHTED CHURN AGAINST A MANEUVERING TARGET (U)

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FOREWORD

This work was performed in the Warfare Analysis Department of the Naval Weapons Laboratory at the request of the Naval Ship Systems Command (PMS-393), reference 1. It concerns an evaluation of a method of passive submarine ranging developed by Dr. D. C.Bossard of Daniel Wagner Associates, reference 2, and some follow-on investigations related to the material in the reference.

The authors had excellent support from R. T. Bevan of the Naval Weapons Laboratory who generated the FORTRAN programs.

This report has been reviewed by R. A. Hodnett, Cdr. USN.

Released by:

R I Rossbacher

R. I. Rossbacher, Head
Warfare Analysis Department

ABSTRACT

In a new sonar bearings-only solution method, Dr. D. C. Bossard of Daniel Wagner Associates achieved quite spectacular reduction in range errors on a zigging target, one-sixth those of the usual (unweighted) CHURN method. His method yields time-corrected range (value at time when expected error is least) and weights observations according to assumed zig statistics. Bossard also advocates extrapolating favorably-chosen time-corrected ranges to obtain present range.

We find that the CHURN, with weights equivalent to Bossard's, achieves equally small time-corrected range errors, and errors at solution time one-third those of the usual CHURN. Random bearing noise, however, seriously degrades solutions using Bossard weights, even without zigs, in which case the unweighted CHURN is optimum. For combinations of zigs and bearing noise, optimum combined weighting functions exist.

Unsuccessful attempts were made to use data available to the tracking ship, e.g. autocorrelation of solution residuals, for selecting optimum weighting. Autocorrelation was also probed for zig detection clues without success.

Results obtained by extrapolating pairs of time-corrected range to present time were about equally as good as from single solutions using the same data.

We conclude that Bossard's important contribution is to show the effectiveness of appropriate statistical weighting.

Further efforts should deal with on-board methods for optimizing weights, and the benefits of weighting for other error sources (bearing bias, own ship position). The results would be applicable not only to the CHURN, but also to the newer optimal filter methods.

CONTENTS

I. INTRODUCTION

The proposal by Wagner Associates to the Naval Ordnance Systems Command (ORD-0521B), reference 2, based on the work of Dr. D. C. Bossard, concerns passive bearings-only ranging on maneuvering targets. An example is given, using synthetic noise-free data, in which Bossard's solution method shows range errors at best range time only one-sixth as large as the corresponding errors of the CHURN method as generally implemented.

We have verified Bossard's results by duplication. In addition we have obtained equally good results with the CHURN modified by the application of statistical weighting equivalent to that used by Bossard. The conclusion is therefore drawn that the power of Bossard's method lies, not in his novel analysis using bearing quadruples, but in weighting observations appropriately for the process (maneuver in this case) which generates the errors.

In the Bossard method a moving window of fixed time span is used. A four-bearing range equation is derived which includes an explicit error term. The derivation continues by formally summing all such equations obtained from suitable sets of bearing quadruples drawn from the window, weighting each addend in accordance with the covariance of its error term. Finally, the equation obtained by summing is modified by adjusting the time parameter so that those terms involving $\dot{v}_{TI}$ (target speed in the line of sight) offset each other. Thus an equation for the time-corrected range for the window is obtained.

The error term appearing in Bossard's four-bearing range equation is essentially a function of $\dot{E}$, the time derivative of the residual E dealt with in the CHURN (E is cross-range miss distance of observed bearing line from estimated target position). As discussed more fully later in this report, the zig strategy assumed in the geometry used here and also by Bossard in reference 2 generates an exponential covariance for $\dot{E}$. Weighting appropriate to this covariance was used by Bossard to obtain his favorable results, and by us in duplicating his results. Appendix A contains the computing algorithm used at the Naval Weapons Laboratory for this purpose.

We have derived the covariance for E which corresponds to the same zig strategy (see part III). This covariance turns out to be non-stationary and correlated, therefore much different from the stationary, uncorrelated statistics usually assumed in applying CHURN. Use of the appropriate covariance in CHURN makes the time-corrected range solutions agree rather closely with those of Bossard's method.

Weighting in accordance with the covariance generated by a zigging target evidently has some value even for certain quite different error distributions, as for example, from the sinuous target motion also tested. When the source of errors includes uncorrelated bearing noise, however, solutions are badly degraded.

II. WEIGHTED CHURN AND TIME-CORRECTED RANGE

The CHURN method estimates four target parameters

a_1 and a_3, east and north components of speed

a_2 and a_4, east and north coordinates at $t = 0$

which are assumed to be constant during the solution data span. From these four parameters, along with own ship motion, all of the familiar target parameters such as range are easily derived. The CHURN solution minimizes the weighted sum of squares of the residual

$$E_i = (a_1 t_i + a_2)\cos B_i - (a_3 t_i + a_4)\sin B_i - x_{oi}\cos B_i + y_{oi}\sin B_i$$

Of symbols appearing here and in Figure 1,

$R_i(a)$ is range at time t_i computed from $a_1 \dots a_4$ and x_{oi}, y_{oi}

B_i is observed bearing at time t_i

x_{oi}, y_{oi} are own ship coordinates at time t_i.

Inspection of Figure 1 shows that E_i is the miss distance of the observed bearing line from the computed target position. The set of equations for all of the E_i $(i = 1, 2 \dots n)$ can be represented in matrix form

$$\begin{bmatrix} E_1 \\ \vdots \\ E_i \\ \vdots \\ E_n \end{bmatrix} = \begin{bmatrix} t_1\cos B_1 & \cos B_1 & -t_1\sin B_1 & -\sin B_1 \\ & \vdots & & \\ t_i\cos B_i & \cos B_i & -t_i\sin B_i & -\sin B_i \\ & \vdots & & \\ t_n\cos B_n & \cos B_n & -t_n\sin B_n & -\sin B_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} - \begin{bmatrix} x_{o1}\cos B_1 - y_{o1}\sin B_1 \\ \vdots \\ x_{oi}\cos B_i - y_{oi}\sin B_i \\ \vdots \\ x_{on}\cos B_n - y_{on}\sin B_n \end{bmatrix}$$

and, with obvious substitutions, in the more compact matrix notation

$$E = Aa - d$$

For minimum variance-covariance of a, two conditions need to be met:

$$E^T WE = \text{minimum over } a$$

$$W = \text{constant} \times (C^E)^{-1}$$

where C^E is the covariance of E. Equations expressing the first condition are obtained by setting

$$\frac{\partial}{\partial a_i}(E^T WE) = 0 \qquad i = 1, 2, 3, 4$$

then substituting for E, and carrying out the differentiation. The resulting four normal equations can be written

$$\psi a = g$$

where $\quad \psi \triangleq A^T WA$ and $g \triangleq A^T Wd$

If the residuals E are stationary and uncorrelated, as has usually been assumed in previous applications of the CHURN, then the C^E appearing in the second condition above is

$$C^E = \sigma_E^2 \, I$$

from which

$$W = \text{constant} \times (\sigma_E^2 \, I)^{-1} = I,$$

merely the identity matrix. We refer to the CHURN thus used as the unweighted CHURN. Other assumptions about the statistics of E, leading to a C^E which is a full matrix, are discussed in Section III.

Time-Corrected Range

For any CHURN solution, there is a time, t*, when the sum of variances of target coordinates is minimum (nearly equivalent to saying the expected range error is minimum):

$$\epsilon\left[(\delta a_2 + t\,\delta a_1)^2 + (\delta a_4 + t\,\delta a_3)^2\right] = \text{minimum over } t$$

in which error quantities are distinguished by a prefixed δ. Evaluation of this equation is made possible by a general property of least-square normal equations: if

$$\psi a = g$$

represents the normal equations, then the covariance of the vector a is

$$\epsilon(\delta a \, \delta a^T) = \psi^{-1}$$

Using this relation to expand the expression to be minimized, we obtain

$$t^2(\psi_{11}^{-1} + \psi_{33}^{-1}) + 2t(\psi_{12}^{-1} + \psi_{34}^{-1}) + \psi_{22}^{-1} + \psi_{44}^{-1} = \text{minimum over } t$$

and setting the time derivative equal to zero yields

$$t = -(\psi_{12}^{-1} + \psi_{34}^{-1})/(\psi_{11}^{-1} + \psi_{33}^{-1}) \triangleq t^*$$

The corresponding range, the "time-corrected range," is

$$R^* = R(a,t^*) = \left[(a_1t^* + a_2 - x_o^*)^2 + (a_3t^* + a_4 - y_o^*)^2\right]^{\frac{1}{2}}$$

The asterisk designates values corresponding to $t = t^*$. Since ψ^{-1} is available as a by product of the straight-forward CHURN process, the additional computation needed for time-corrected range is trivial. The accuracy of this range is considerably better than for range at solution time. Since the time-corrected range is older, however, and since the other solution parameters are not improved, its usefulness is controversial.

III. COVARIANCE OF E

As previously stated, the Bossard method makes the assumption that the time derivative $\dot{E}$ of the residual E is exponentially autocorrelated. According to the target motion used by Bossard in reference 2, zig times are to be randomly drawn from a Poisson distribution such that the probability of no zig within time interval t_i to t_j is

$$e^{-|t_i - t_j|/t_m} \tag{3-1}$$

where t_m is the mean time between zigs. The new course is to be randomly selected from a rectangular distribution extending $\pm \delta C_T$ max degrees about a mean course. With these conditions, Appendix B shows that the target course deviation has a covariance matrix

$$\epsilon(\delta C_T \, \delta C_T^T)_{ij} = \sigma_{CT}^2 \, e^{-|t_i - t_j|/t_m} \; (\text{degrees})^2 \tag{3-2}$$

where $\sigma_{CT}^2 = \frac{1}{3}(\delta C_T \text{ max})^2$. If the bearing rate is small, $\dot{E}$ is nearly proportional to δC_T. Thus the covariance matrix for $\dot{E}$ should have the same form:

$$C^{\dot{E}} \triangleq \epsilon(\dot{E}\dot{E}^T) = C_{00}^{\dot{E}} \, e^{-|t_i - t_j|/t_m} \tag{3-3}$$

In order to use equivalent weights in the CHURN, which minimizes E^TWE rather than $\dot{E}^TW\dot{E}$, the corresponding covariance of E is needed.

This has been derived as outlined here (given in more detail in Appendix C):

$$C_{ij}^{E} \triangleq \epsilon(E_i E_j) = \epsilon\left[(E_o + \int_0^{t_i} \dot{E}\, dt)(E_o + \int_0^{t_j} \dot{E}\, dt)\right] \tag{3-4}$$

$$= \epsilon\left[E_o^2 + E_o(E_i + E_j - 2E_o)\right] + \int_0^{t_i}\int_0^{t_j} \epsilon\left[\dot{E}(t)\dot{E}(t')\right] dt\, dt'$$

$$= -C_{oo}^{E} + C_{io}^{E} + C_{oj}^{E} + \sigma_{\dot{E}}^{2} \int_0^{t_i}\int_0^{t_j} e^{-|t' - t|/t_m}\, dt\, dt' \tag{3-5}$$

After integration one obtains as final result

$$C_{ij}^{E} = -C_{oo}^{E} + C_{io}^{E} + C_{oj}^{E} + \sigma_{\dot{E}}^{2} t_m^2 \left[-e^{-|t_i - t_j|/t_m} + e^{-|t_i|/t_m} + e^{-|t_j|/t_m} - 1 + (|t_i| + |t_j| - |t_i - t_j|)/t_m\right] \tag{3-6}$$

The functions $C_{io}^{E}(t_i)$, $C_{oj}^{E}(t_j)$, and the constant C_{oo}^{E} which result from integration are completely arbitrary as far as the original specification of $\dot{E}$ is concerned, but must be selected so that C^E retains the general properties of covariance/autocorrelation matrices: C^E should be positive definite as well as symmetric. Therefore C_{io}^{E} and C_{oj}^{E} should be identical functions of t_i and t_j, respectively. For simplicity, these restraints were met by arbitrarily setting

$$C_{io}^{E} = C_{jo}^{E} = C_{oo}^{E} = \sigma_{\dot{E}}^{2}\, t_m^{2}$$

Again for simplicity, the factor $\sigma_E^{.2}\ t_m^2$ was taken as unity, since for weighting purposes a constant factor on the covariance has no effect. The result is the expression used in many of the tests:

$$C^E \text{ Simple} \triangleq -e^{-|t_i-t_j|/t_m} + e^{-|t_i|/t_m}$$

$$+ e^{-|t_j|/t_m} + (|t_i| + |t_j| - |t_i-t_j|)/t_m \qquad (3\text{-}7)$$

The origin of time was taken at the middle of the data window.

Seeking a more logical way of selecting the arbitrary functions C_{io}^E, C_{oj}^E and constant C_{oo}^E, we have derived a covariance $\widetilde{C}_{ij}^E$ of deviations from the mean $\overline{E}$ over any selected time interval $t_a - \tau$ to $t_a + \tau$. The derivation (given in full in Appendix D) starts with

$$\overline{E} \triangleq (1/2\tau) \int_{t_a-\tau}^{t_a+\tau} E_i \, dt_i \triangleq 0 \qquad (3\text{-}8)$$

It follows that

$$\epsilon(E_j\overline{E}) = (1/2\tau) \int_{t_a-\tau}^{t_a+\tau} (E_iE_j)dt_i = 0 \qquad (3\text{-}9)$$

$$0 = (1/2\tau) \int_{t_a-\tau}^{t_a+\tau} C_{ij} \, dt_i \qquad (3\text{-}10)$$

After substituting from equation (3-6) into (3-10) and integrating, it is possible to solve for C_{io}^E, C_{oj}^E and C_{oo}^E.

The complete covariance expression is then

$$\mathcal{C}^E_{ij} = \sigma_{\dot{E}}^2 \, t_m^2 \Big\{ -e^{-|t_i-t_j|/t_m} - |t_i-c_j|/t_m$$

$$-(t_m/2\tau) \, e^{-\tau/t_m} \left[e^{(t_i-t_a)/t_m} + e^{-(t_i-t_a)/t_m} \right] + (t_i-t_a)^2/2t_m\tau$$

$$-(t_m/2\tau) \, e^{-\tau/t_m} \left[e^{(t_j-t_a)/t_m} + e^{-(t_j-t_a)/t_m} \right] + (t_j-t_a)^2/2t_m\tau$$

$$+ (t_m/\tau) + \tau/3t_m + (t_m^2/2\tau^2) \left[1 - e^{-\tau/t_m} \right] \Big\} \qquad (3\text{-}11)$$

Some computer runs were made with this covariance formula, using for t_a the center of the window, and for τ one-half the window length.

There is some logic in the notion of placing t_a at the end of the data window instead of at the center. This should have the effect of weighting the more recent observations more heavily, possibly yielding more up to date solutions. This idea has not yet been tried.

We were somewhat puzzled to find that certain widely-differing covariance matrices yielded near identical range results, while in other cases a diminutive change to the assumed covariance matrix significantly influenced the results. The former situation is illustrated by C^E_{ij} Simple and $\mathcal{C}^E_{ij}$ for which equations are given above, (3-7) and (3-11). Numerical elements of the covariance matrices discussed here and of the weighting matrices which are their inverses are given in Appendix E. No resemblance is apparent either between covariance matrices or weighting matrices, yet the results obtained were nearly identical. The other situation is illustrated by (C_{ij}^{E} Simple + .002I). Although the diagonal elements of C_{ij}^{E} Simple are thus increased by no more than 0.2%, the corresponding inverse matrix is thereby noticeably changed, and range results were appreciably influenced.

Experiment has also shown that the combination ($\tilde{C}_{ij}^{E}$ + .002I) produced ranges nearly identical to those of (C_{ij}^{E} Simple + .002I), RMS values differing less than 5 yards in cases tested.

IV. <u>TEST CASES</u>

The tests here reported used mainly the geometry shown in Figure 2, this being the same as used by Bossard in reference 2. According to the reference, the target path in this geometry was constructed by the algorithm described in section III, using t_m = 15 minutes as the mean time between zigs, and selecting each new course from a rectangular distribution lying within ±40° of the mean course, 142°. Own ship retained an unchanged schedule of zigs throughout all tests, and both ships maintained speed constant at approximately 10 knots.

Solutions were computed for a moving data span or "window" usually 20 minutes in length, which included 11 bearing samples taken at odd-numbered minutes. Thus the first solution came at 21 minutes, followed by new solutions at 2-minute intervals.

In order to test certain hypotheses, variations in the above conditions and solution methods were introduced. These included the straight target path and sinuous target path shown in Figure 3, bearings with noise, and a longer data window.

V. DISCUSSION OF RESULTS

1. Churn vs. Quadruples

The initial question attacked was why the quadruple method proposed in reference 2 yielded better ranges than the CHURN for the synthetic case used for demonstration (zigging target, noise-free bearings). Several ideas were tried before equal performance was achieved.

In the quadruple method, the single variable range is optimized. In CHURN, on the other hand, the "time-corrected range" corresponds to the minimum sum of variances of the two variables x and y. It seemed possible that if by rotation of coordinates the range vector was made nearly parallel to the y-axis, and if then the variance of y alone was minimized, that a better range value would be obtained. No significant gain was realized, however, as can be seen by comparing columns 3 and 4, first row, of Table 1.

The next experiment was intended to try in the CHURN a weighting matrix equivalent to that used in the quadruple method. Through an initial misunderstanding of this weighting, weights corresponding to

$$C^E_{ij} = e^{-(t_i-t_j)/t_m}$$

were applied to the CHURN. Table 1 exhibits the poor results thus obtained (column 5).

Another experiment used the same weights divided by the range. It is well known that CHURN produces biased estimates because the residual E is scaled by the range. In a synthetic problem in which true ranges are known a priori, some of the bias can be removed by dividing out the range. Column 6 of Table 1 shows the improvement to be neglibible.

Further examination of Bossard's quadruple formulation revealed that the assumed exponential (Poisson) distribution applies essentially to $\dot{E}$, rather than to E. The derivations of section III and Appendices B, C yielded the corresponding covariance matrix needed to apply this assumption in the CHURN. As shown in the first row, last column of Table 1, the range errors thus obtained with the CHURN are even slightly smaller than those of the quadruple method. It was concluded that the merit observed in the quadruple method arises from the weighting, rather than from geometrical properties. Subsequent tests, using the CHURN, were directed toward learning the effect of this weighting and its variations under differing conditions.

Although achieving a substantial advantage for a maneuvering target in the absence of bearing meansurement noise, the type of weighting used in the quadruple method performed poorly when noise was present. In order to separate the effect of bearing noise from the effect of maneuver, tests were performed on a straight-running target with unweighted CHURN and with weighted CHURN. The second row in Table 1 shows that the unweighted CHURN performs better for this case.

TABLE 1

RMS RANGE ERROR - YARDS

TARGET MANEUVER	RMS BEARING NOISE	QUADRUPLE		UNWEIGHTED 3	UNWEIGHTED ROTATED 4	WEIGHT I 5	WEIGHT I ÷ R 6	WEIGHT II 7
		BOSSARD 1	NWL 2					
ZIG	0	931 --	1060 --	4110 6720	4261 6722	3826 5539	3794 5427	700 1915
STRAIGHT	0.1°			595 993				6369 5599

Notes:

1. Upper figure time-corrected range, lower figure range at solution time.
2. Noise-free cases are averaged over the 15 windows of a single run. Cases with noise are averaged over three runs, 15 windows each.
3. Weight I corresponds to $C^E = e^{-|t_i-t_j|/t_m}$
4. Weight II corresponds to $C^E = -e^{-|t_i-t_j|/t_m} + e^{-|t_i|/t_m} + e^{-|t_j|/t_m} + (|t_i| + |t_j| - |t_i-t_j|)/t_m \triangleq C^E$ Simple

2. Variation of Covariance Parameters

We desired to investigate whether adjusting the parameters of the assumed covariance upon which weighting is based would bring about any improvement, especially in the presence of bearing noise. The second line in Table 1 indicates that the Bossard weighting performs poorly when uncorrelated bearing noise is present. It seemed reasonable that other values of t_m might work at least as well, since only three zigs at most occurred within any window, and such a zig pattern could arise from a wide range of t_m values with almost equal probabilities. In addition, the derivation process (Appendix C) contributes to the covariance of E an undetermined constant C_{oo} and undetermined functions C_{io}, C_{jo}. A series of tests was performed to compare results with variations in these parameters, and the range errors are given in Table 2. The upper half of the table shows results obtained by changing $(C^E_{io} + C^E_{oj} - C^E_{oo})$, e.g. adding a uniform constant to each element in the covariance matrix. The effect on range errors is insignificant, with or without noise.

The lower part of Table 2 indicates the effect of varying t_m. Performance in the no-noise case deteriorates, but in the more realistic case of bearings with noise, very substantial improvement can be obtained by reducing t_m to 0.2 - 0.1. The logic underlying this result is only partly clear. The shorter mean time to zig implies less correlation between observations, and thus corresponds better to the uncorrelated bearing noise portion of the residual E.

TABLE 2

VARIATION OF t_m AND $(C^E_{io} + C^E_{oj} - C^E_{oo})$

$C^E_{io} + C^E_{oj} - C^E_{oo}$	t_m(min)	RMS Range Error: No Noise, At Solution Time	RMS Range Error: No Noise, Time-Corrected	RMS Range Error: Noise $\sigma_N = 0.1°$, At Solution Time	RMS Range Error: Noise $\sigma_N = 0.1°$, Time-Corrected
0.2	15.0	1914	699	6021	6715
1.0	"	1915	700	6013	6715
5.0	"	1921	709	6040	6716
1.0	28.6	2003	795		
"	15.0	1915	700	6013	6715
"	7.0	1995	725		
"	2.18			4140	3471
"	1.66			3786	2780
"	1.24			3614	2288
"	0.87	4595	2564	3691	2104
"	0.87			4263	1963 (10)
"	0.67			3849	2084
"	0.43			4612	2348 (10)

Target maneuver: zigs

$$C^E = -e^{-|t_i - t_j|/t_m} + e^{-|t_i|/t_m} + e^{-|t_j|/t_m} + (|t_i| + |t_j| - |t_i - t_j|)/t_m$$

$$+ C^E_{io} + C^E_{oj} - C^E_{oo}$$

(10) RMS of 10 noise samples. Other entries RMS of 3 noise samples.

3. Covariance for Combined Error Sources

Derivation of the covariance as for the results of Table 2 accounts only for residuals arising from zigs, whereas the covariance commonly used in the CHURN (proportional to an identity matrix) assumes residuals arising only from Gaussian bearing measurement noise. When both sources contribute to the residuals, it is reasonable to add their covariance contributions. The sum, expressed in a form to make the units of both contributions consistent, is

$$\text{Combined covariance} = \sigma_{\dot{E}}^2 t_m^2 \, C^E\text{Simple} + \sigma_{EB}^2 \, I$$

where C^E Simple is defined by equation (3.7), σ_{EB}^2 is the variance of E caused by bearing measurement noise, and I is the identity matrix of the same order as C^E. Rearranging,

$$(\text{Combined covariance})/(\sigma_{\dot{E}}^2 t_m^2) = C^E \text{ Simple} + \sigma_{EB}^2/(\sigma_{\dot{E}}^2 t_m^2) \, I$$

from which we define the equivalent terms

$$C^E \text{ Combined} = C^E \text{ Simple} + \alpha I$$

We have performed a number of CHURN tests to determine whether the C^ECombined leads to significant improvement. Since the factors upon which α depends would not be known with precision in practical situations, we also wished to determine whether its value is critical.

Figures 4, 5, 6 show the effects of variation in α on range errors for three types of target motion. All cases include uncorrelated bearing measurement noise. The RMS range errors have been plotted against αt_m^2 in order to normalize for the several values of t_m included. Figure 4 verifies an expected result, that for a straight-running target the unmodified identity weighting ($\alpha = \infty$) is optimum.

Figure 5 applies to the zigging target used in reference (2). This graph confirms that a proper blend of C^E Simple and I is significantly better than either alone for this target in the presence of bearing noise, except when t_m is very small. It also indicates that α may vary by a factor of three either way from the optimum without serious degradation of the range solution.

Figure 6, for a sinuous target motion and bearing noise, also shows advantage for the C^E Combined, even though the statistical distribution of the residual E generated by this maneuver is entirely different than that generated by a zigging target. The factor α is again noncritical.

Returning to consider Figure 4 again along with Table 2, it appears that as much is gained by reducing t_m as by using C^E Combined. Figure 7, however, supports the use in the absence of noise of a value near the theoretical one, t_m = 15 minutes. The conclusion to be drawn is, that the theoretically appropriate values of t_m and α work at least as well as any, but when the effect of random noise is large, certain other values also work as well. A more specific characterization at this stage seems hazardous.

4. Tactical Selection of Covariance Parameters

It is clear from Figures 4 through 7 that for best results from the CHURN, the assumed covariance parameters t_m and α should be adjusted to suit the noise and target maneuver. Since the a priori information on target maneuver available to the tracking ship would be limited, a method of adjustment based on preliminary data analysis is desirable. We have examined several schemes for selecting the parameters, with doubtful or negative results. One method consisted of comparing the variance of bearing residuals based on the same data but with different assumed functions for the covariance of E. It was hoped that small residuals would correlate closely with small range errors. In Figure 8, the variance of residuals is plotted against RMS error of time-corrected range, and in Figure 9 against RMS error of solution-time range. Points plotted in these figures are coded to distinguish three combinations of zig and/or bearing noise, and for each combination a variety of covariance functions is represented. While some correlation is noticeable between bearing residuals and range errors, the relation is not as consistent as one desires as a basis for choice.

5. Zig Detection by Autocorrelation of Bearing Residuals

Unfavorable results were obtained from our attempts to detect zigs by means of the autocorrelation of bearing residuals. The principle involved is that the shape of the autocorrelation curve estimated from the sample,

$$R(\tau)_{sample} \triangleq [\text{mean } \delta B(t)\, \delta B(t + \tau)]/[\text{mean } \delta B(t)\, \delta B(t)]$$

should be different for zigging and nonzigging targets. If one assumes that bearing residuals should have the same covariance (except for a scale factor) as the cross-range residual E, then for a zigging target with bearing noise the expected $R(\tau)_{sample}$ would be the theoretical curve drawn in Figure 10 on the left side. This curve was computed from the covariance

$$\widetilde{C}^E + .002\ I$$

For a non-maneuvering target with noise, the expected value of $R(\tau)_{sample}$ would be unity at the origin and zero every where else.

Figure 10 exhibits the empirical $R(\tau)_{sample}$ points obtained for two sets of three runs. Both sets are identical in conditions except that one has a zigging target while the other has a straight-running target. The "random" sequence of bearing noise values is the same for each set. It does not appear that any clearcut characteristic distinguishes the zig cases from the straight path cases. Furthermore, each curve is an average over 49 minutes of data, which is generally too long to wait for the basis of a shipboard decision. With a lower noise level a better result would possibly be obtained, but the level assumed is not unreasonable. The value $\sigma_B = 0.1°$ applies to generated bearings two minutes apart, corresponding to a mean of sixty or so raw bearings.

6. Time-Corrected Solutions vs. Longer Windows

Reference (2) proposes extrapolation from a sequence of time-corrected ranges to obtain range at present (or desired) time. It gives examples of time-correction applied to CHURN to support the assertion that the line joining two time-corrected ranges from suitably chosen windows can be extended to estimate present range at least as well as a single solution covering the entire interval. The advantage is said to be greater if the tracking ship executes a lag-lead-lag series of course legs.

Our tests of this method employed different conditions than those of reference (2). The most significant differences are that we used the zigging target of Figure 2 (instead of a straight path), a different noise level, and weights corresponding to

$$C^E\text{Combined} = C^E\text{Simple} + .002\ I$$

in the CHURN (instead of the unweighted CHURN). This covariance has, in our other tests, yielded the best overall range results for this geometry and noise level ($\sigma_B = 0.1°$). Time-corrected ranges were obtained for every 20-minute window possible in the run, and solution time ranges for every 28-minute run. The process was repeated for two additional runs with different noise samples. In Figure 11 the solid lines represent errors of time-corrected range plotted against the "best range time" at which the time-corrected range applies; certain points have been labeled with the time of solution.

As predicted by reference (2), the best range time tends to fall near the beginning of a window which spans a lag-lead sequence of tracking ship legs (points labeled 21 or 39), and near the end of a window which spans a lead-lag sequence (29 or 49). Thus the pairs 21 and 29 or 39 and 49 provide favorably long baselines for extrapolation. Circled points terminating the short-dashed lines show the range errors resulting from extrapolation to present time, at 29 or 49 minutes.

Figure 11 also shows the range errors at solution time using 28-minute windows (square points). The 28-minute windows ending at 29 or 49 minutes encompass nearly the same data as the pairs of 20-minute windows used in extrapolation. At 29 and 49 minutes, therefore, a direct comparison can be made between the two methods: extrapolation of time-corrected range using pairs of short windows, versus solution time range using longer windows. In this case the verdict is nearly a tie, but it is evident that the scatter caused by bearing noise would mask the small advantage which might exist for either method.

VI. CONCLUSIONS

We find that the major contribution of Dr. Bossard in reference (2) to passive TMA technology is to demonstrate the effectiveness of appropriate statistical weighting. Our own tests have shown that the CHURN, with equivalent weighting, performs as well as the quadruple method presented in reference (2). The importance of weighting would apply to any type of solution using redundant data.

The weighting used in reference (2) was derived to fit a particular target maneuver strategy, assuming no bearing noise. When random bearing noise is present, however, solutions with this weighting are poor even without zigs; indeed, the unweighted version of CHURN is then optimum. A combined weighting has been found to be best when both noise and maneuver are present. Although the combining proportions are not critical, further work is needed to develop a method for selecting the best weighting on the basis of information available to the tracking ship.

Compared to the unweighted CHURN, the CHURN with weighting derived for a zigging target has been found superior for a sinuous target path, although the residuals generated by sinuous motion have an entirely different statistical distribution. This result leads us to hope that one type of weighting can be used for a variety of target strategies.

Our attempts to distinguish target zigging from straight path by examining bearing error autocorrelation were unsuccessful. This idea would probably work at lower levels of random bearing noise.

In our tests of time-corrected ranges extrapolated to estimate present range, no advantage was observed over solution time range. Since the number of cases tried was only three, the scatter due to noise masked whatever small difference might exist. Time-corrected ranges probably have other applications, since they indeed have smaller variance than solution-time ranges. For instance, information that the target has been approaching since best range time may exist in a form not available to the computer; even if solution time range is wild, the time-corrected range would provide a fairly reliable upper limit.

The statistics of maneuver introduced here in the CHURN can also be introduced in the optimal (Kalman) filter technique now coming into favor. Reference 3 describes a Kalman filter and trials on real data. The formulation given permits statistical variations of target velocity components, i.e. target maneuver, and in some of the trials the filter did assume non-zero statistics for velocity changes. All of the cases tried, however, had a non-maneuvering target. The suboptimal results correspond to our own results with the CHURN (reported herein), when using weighting appropriate to a zigging target on a straight-running target.

VII. AREAS FOR FUTURE WORK

Taking full advantage of weighting requires a method for selecting a near-optimum weighting function, given such information as a tracking ship can possess in the tactical environment. Our few unsuccessful attempts were aimed at extracting a weight selection criterion from bearing data alone. While we believe this not to be hopeless, further studies should consider the use of bearings together with other sensor data in a combined solution method. With the maneuver detection problem thus greatly reduced, appropriate bearing statistics would be available to enhance the combined solution.

In addition to target zigs, other sources of correlated errors exist for which appropriate weighting could improve solutions. Examples are own ship position keeping, and bearing measurement bias. Enough is known about these particular sources so that error modeling would not be difficult. Investigation is needed to determine how large the potential improvement would be, and how to introduce the statistics into a solution.

Inasmuch as the Kalman filter solution method appears soon to become the standard, further innovations should either be implemented in this framework or tested in competition to it with respect to accuracy and computing economy.

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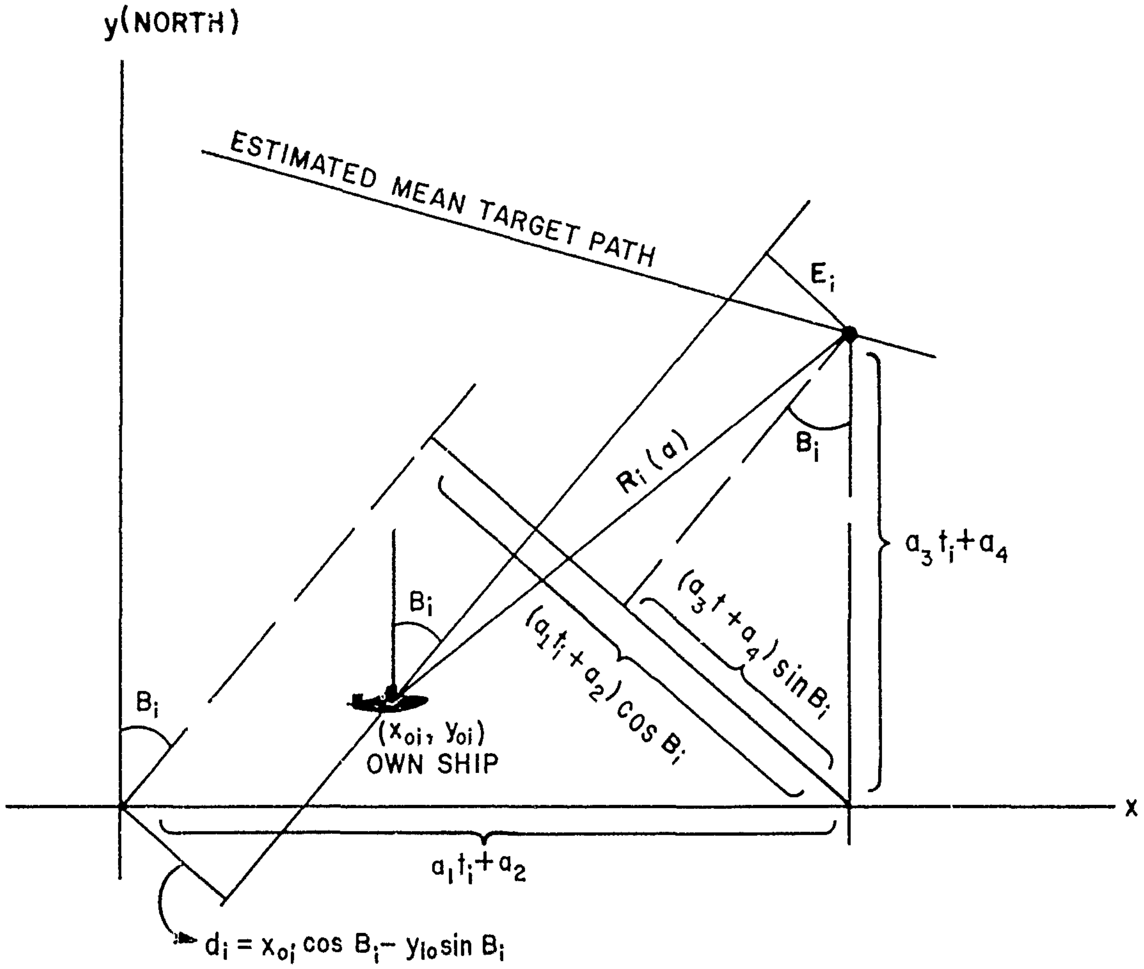
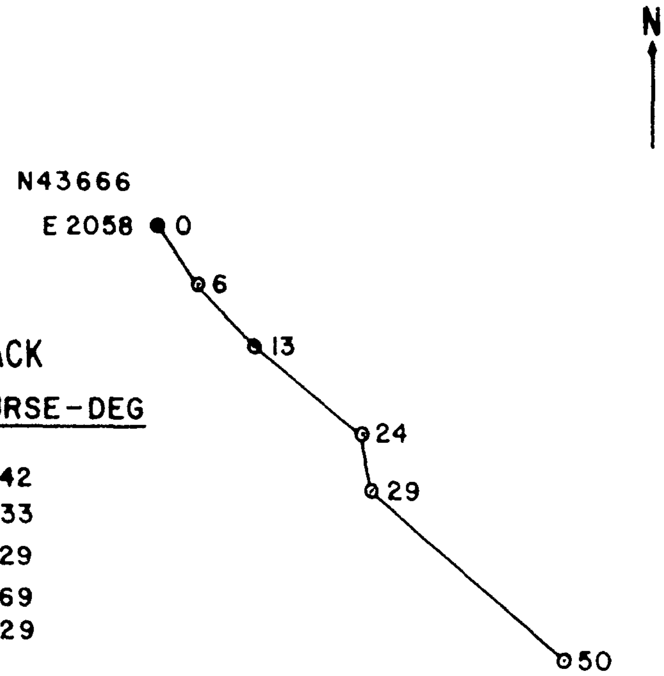
2. Passive Ranging on Maneuvering Targets, Daniel H. Wagner, Associates, Proposal to Naval Ordnance Systems Command, Enclosure (1), 13 January 1970, (CONF)

3. Comparison of Optical Filter Target Motion Analysis Mathematical Models Using Real Fire Control Data (U), Robert W. Palmer, NOLTR 70-182, 11 Jan 1971 (CONF)



CHURN GEOMETRY

FIGURE 1



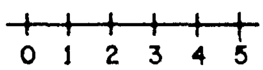
TARGET TRACK

TIME-MIN.	COURSE-DEG
0-6	142
6-13	133
13-16	129
24-29	169
29-50	129

SPEED 9.86 KNOTS

GEOGRAPHIC PLOT OF ENCOUNTER

SCALE (KYD)



SSK TACTIC

TIME-MIN.	COURSE-DEG
0 - 7	298
7 - 8	0
8 - 18	60
18 - 19	0
19 - 30	300
30 - 40	60
40 - 50	300

SPEED 9.86 KNOTS

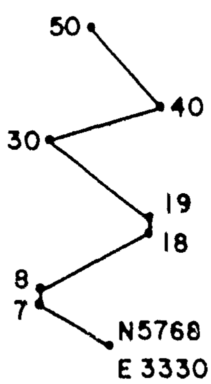
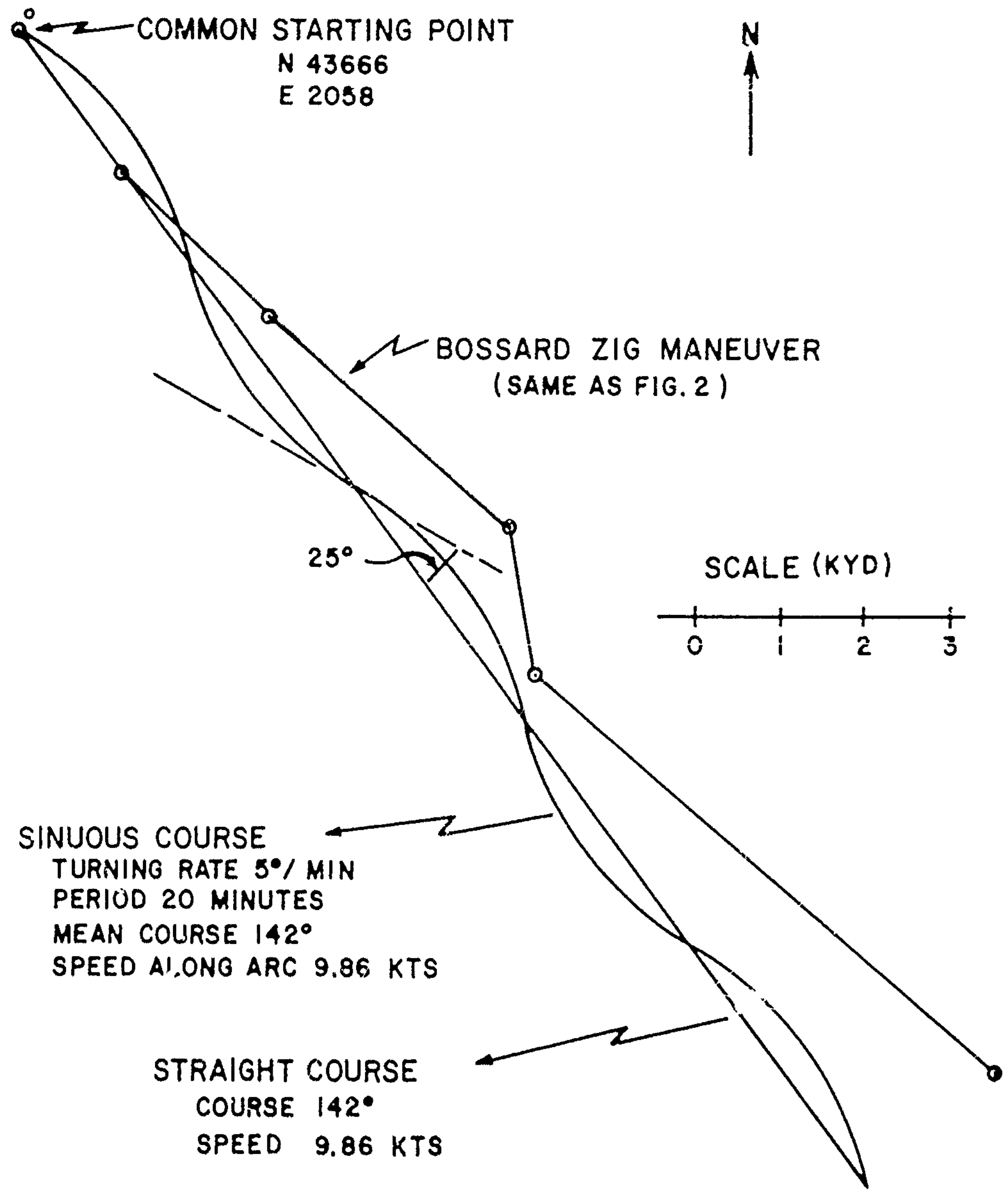


FIGURE 2

UNCLASSIFIED



TARGET TRACKS USED IN TESTS

FIGURE 3

UNCLASSIFIED

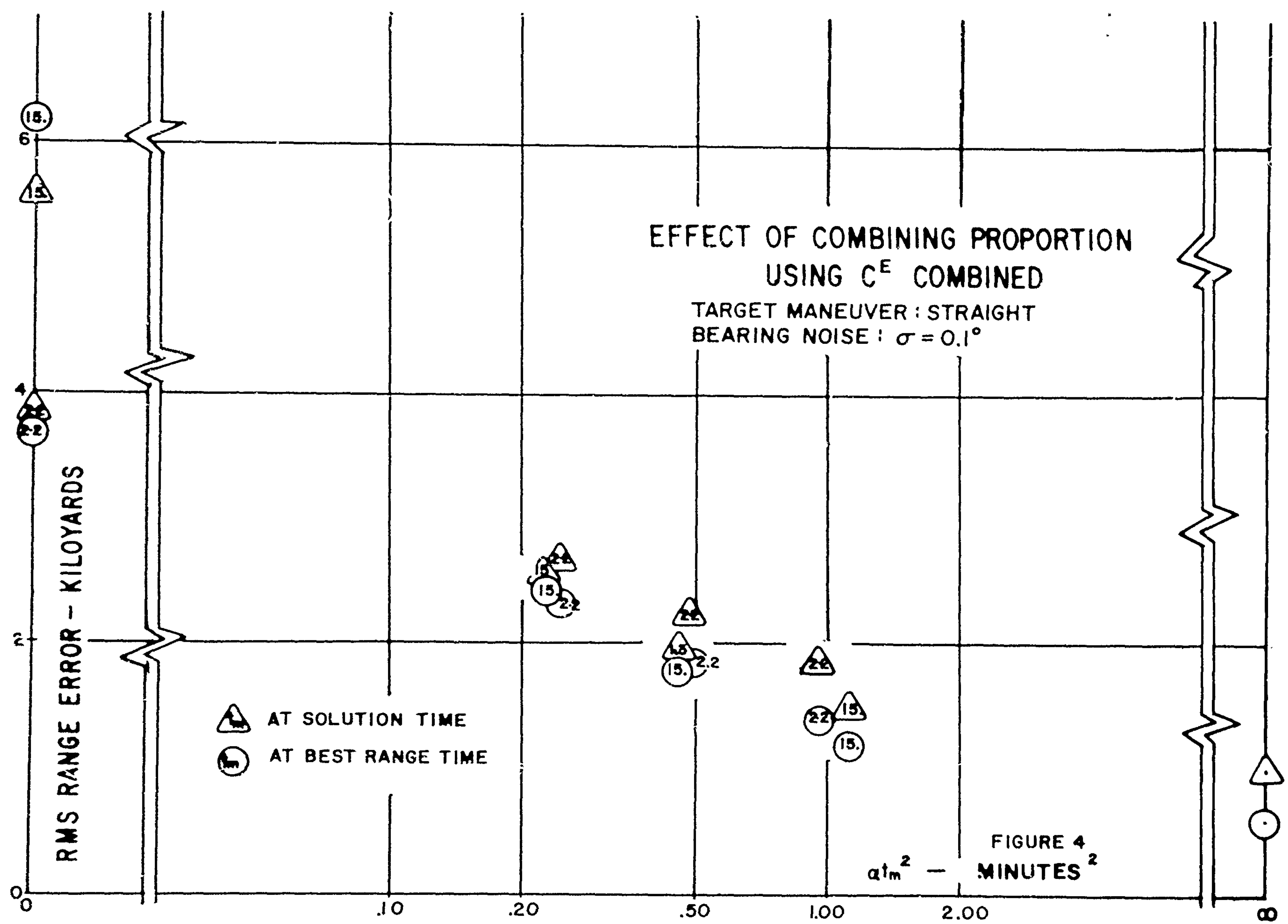

EFFECT OF COMBINING PROPORTION
USING C^E COMBINED
TARGET MANEUVER : STRAIGHT
BEARING NOISE : $\sigma = 0.1°$
RMS RANGE ERROR – KILOYARDS
AT SOLUTION TIME
AT BEST RANGE TIME
at_m^2 — MINUTES^2
0
2
4
6
.10
.20
.50
1.00
2.00
∞
15.
2.2

FIGURE 4

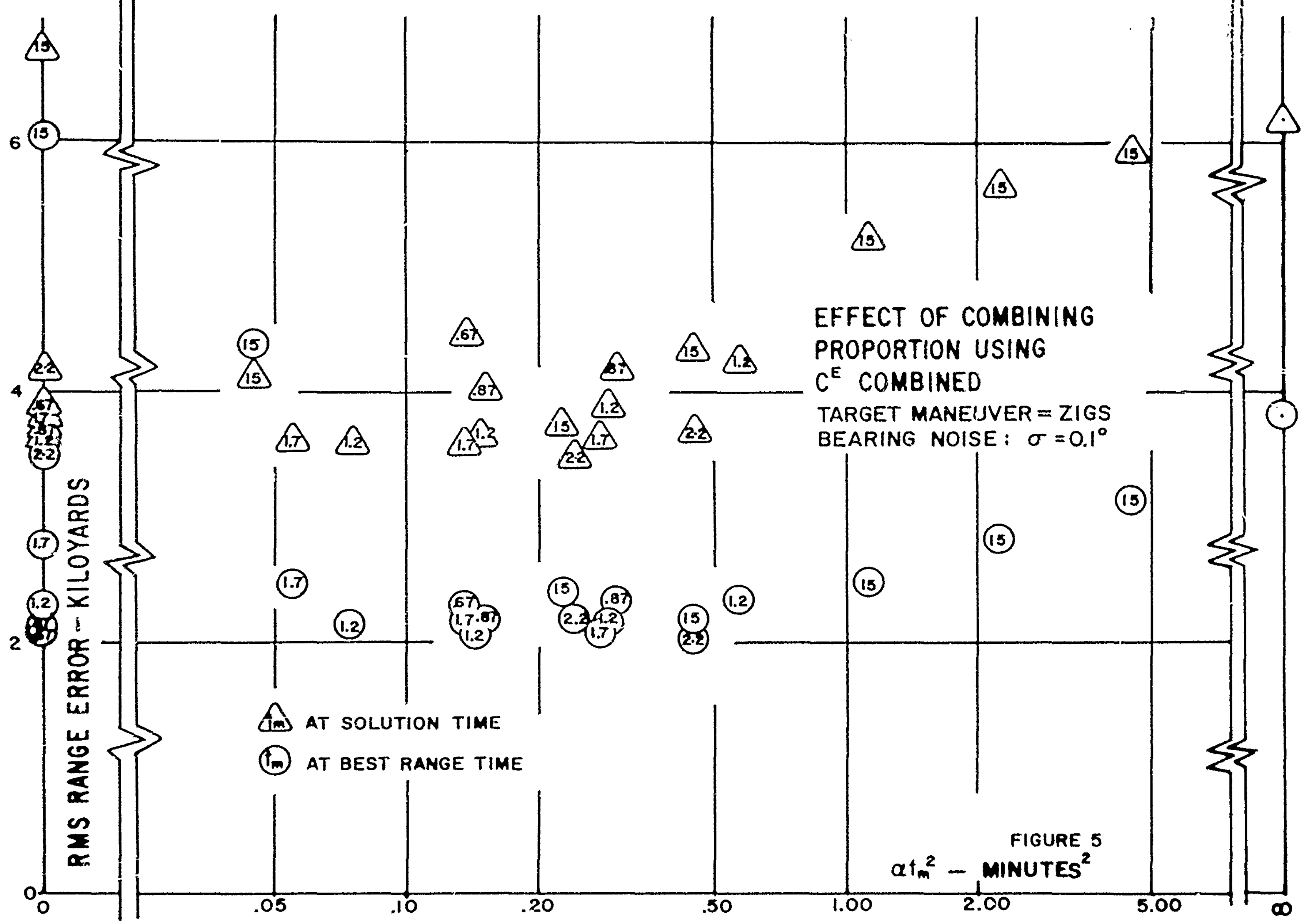

EFFECT OF COMBINING
PROPORTION USING
C^E COMBINED
TARGET MANEUVER = ZIGS
BEARING NOISE : σ = 0.1°
AT SOLUTION TIME
AT BEST RANGE TIME
RMS RANGE ERROR - KILOYARDS
0
2
4
6
0
.05
.10
.20
.50
1.00
2.00
5.00
∞
αt_m^2 — MINUTES^2

FIGURE 5

CONFIDENTIAL

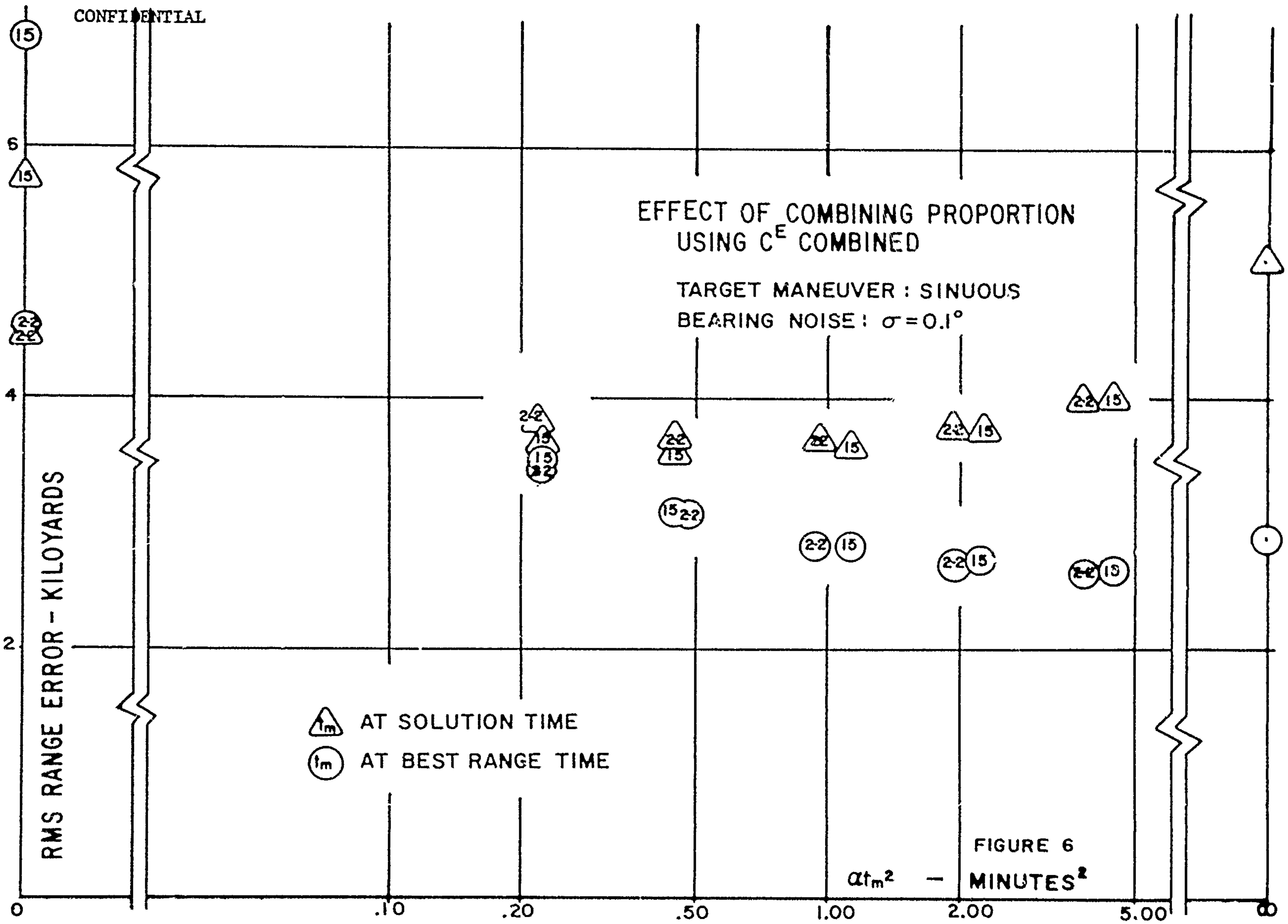


FIGURE 6

CONFIDENTIAL

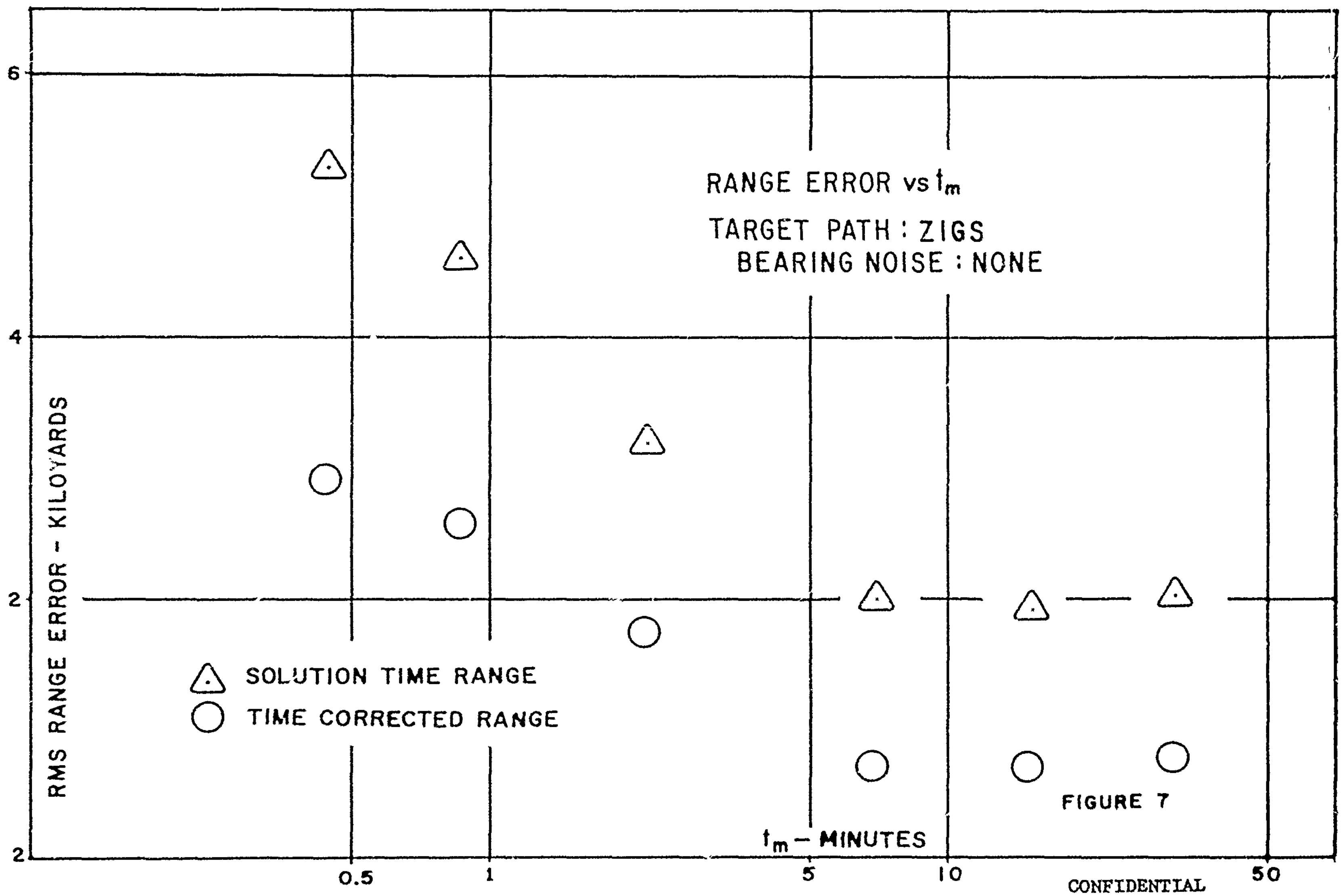

RANGE ERROR vs t_m
TARGET PATH : ZIGS
BEARING NOISE : NONE
SOLUTION TIME RANGE
TIME CORRECTED RANGE
RMS RANGE ERROR - KILOYARDS
t_m — MINUTES
6
4
2
2
0.5
1
5
10
50

FIGURE 7

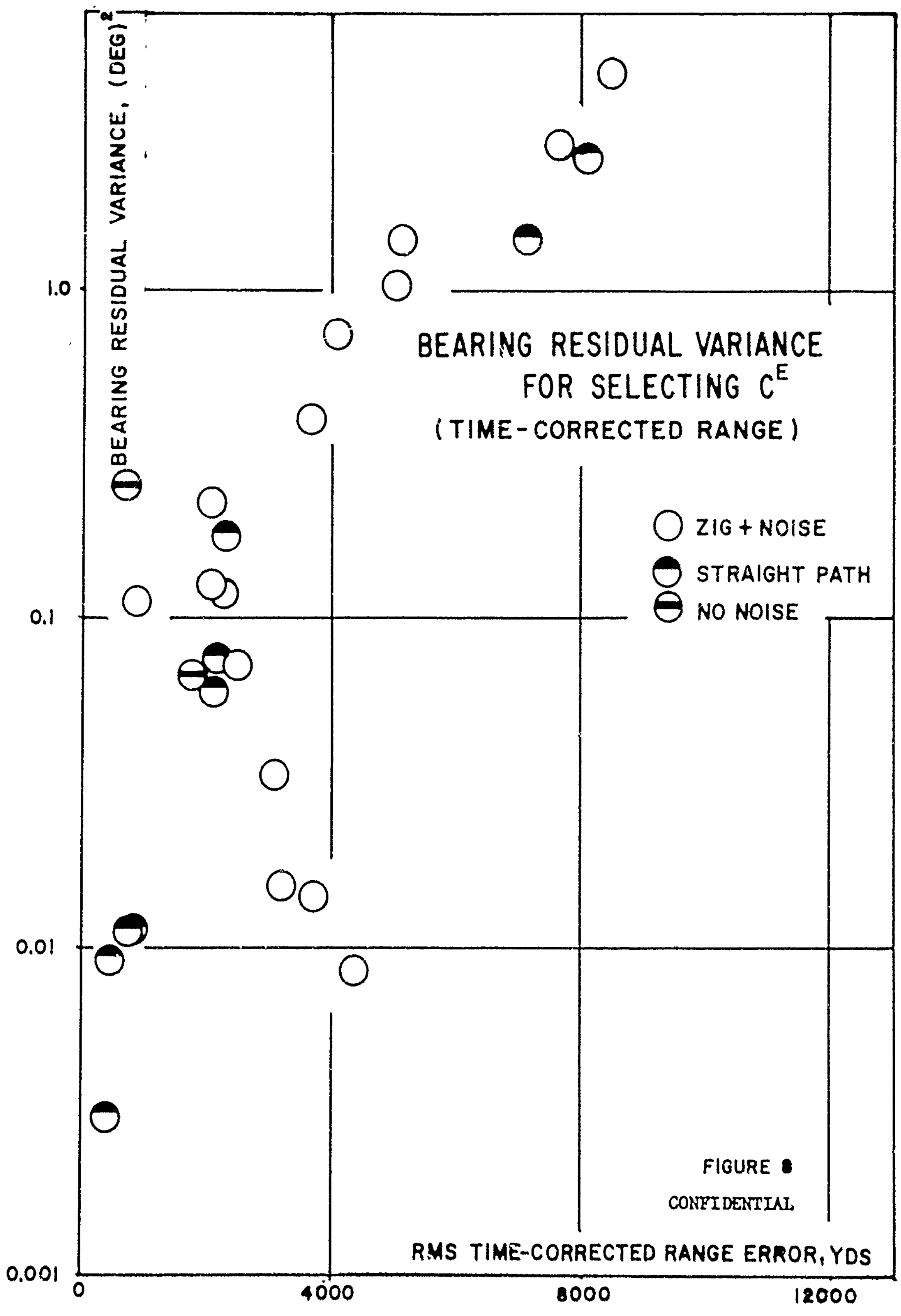

BEARING RESIDUAL VARIANCE
FOR SELECTING C^E
(TIME-CORRECTED RANGE)
ZIG + NOISE
STRAIGHT PATH
NO NOISE
BEARING RESIDUAL VARIANCE, (DEG)^2
RMS TIME-CORRECTED RANGE ERROR, YDS
1.0
0.1
0.01
0.001
0
4000
8000
12000

FIGURE 8

CONFIDENTIAL

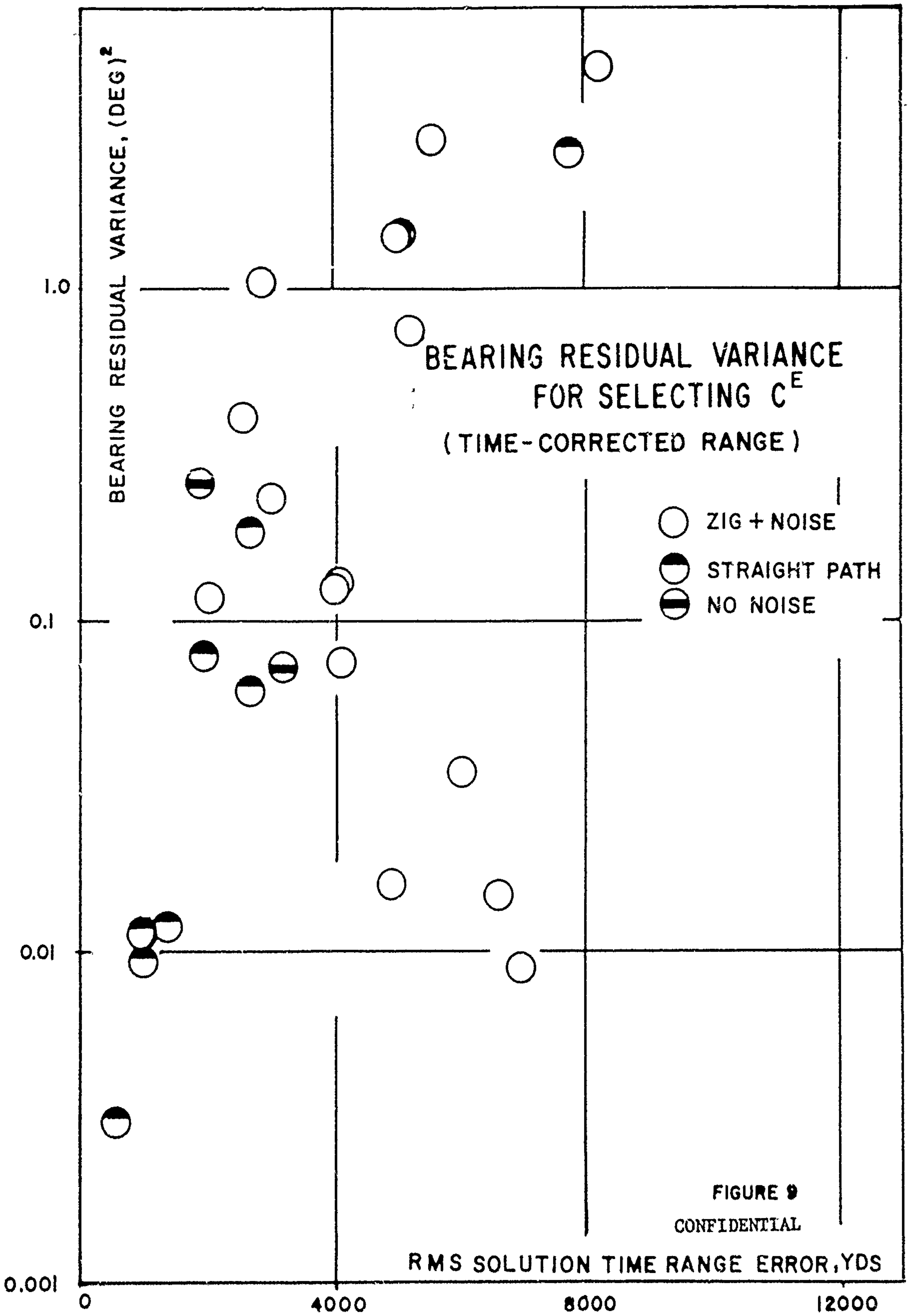


FIGURE 9

CONFIDENTIAL

CONFIDENTIAL

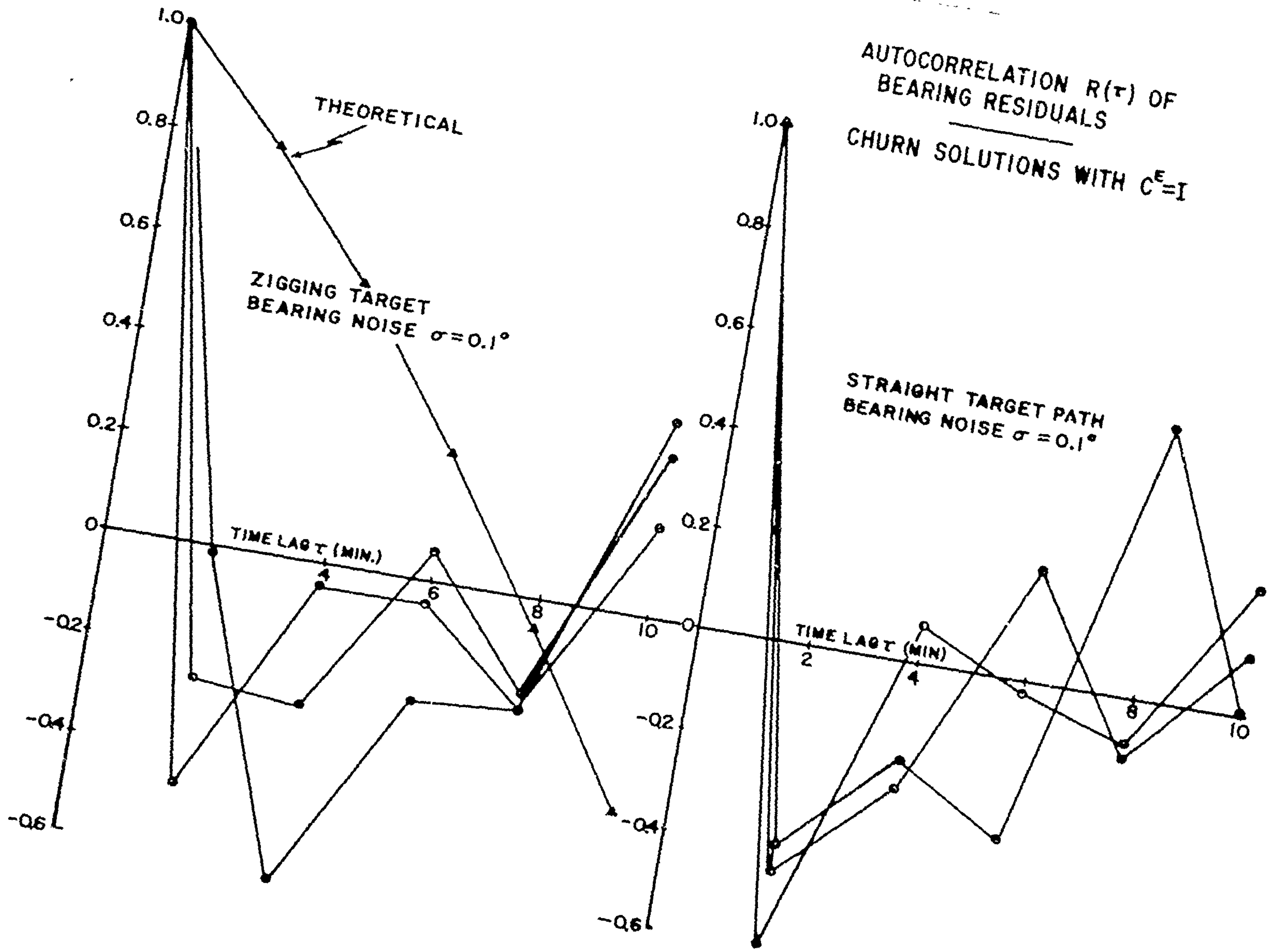


FIGURE 10

CONFIDENTIAL

CONFIDENTIAL

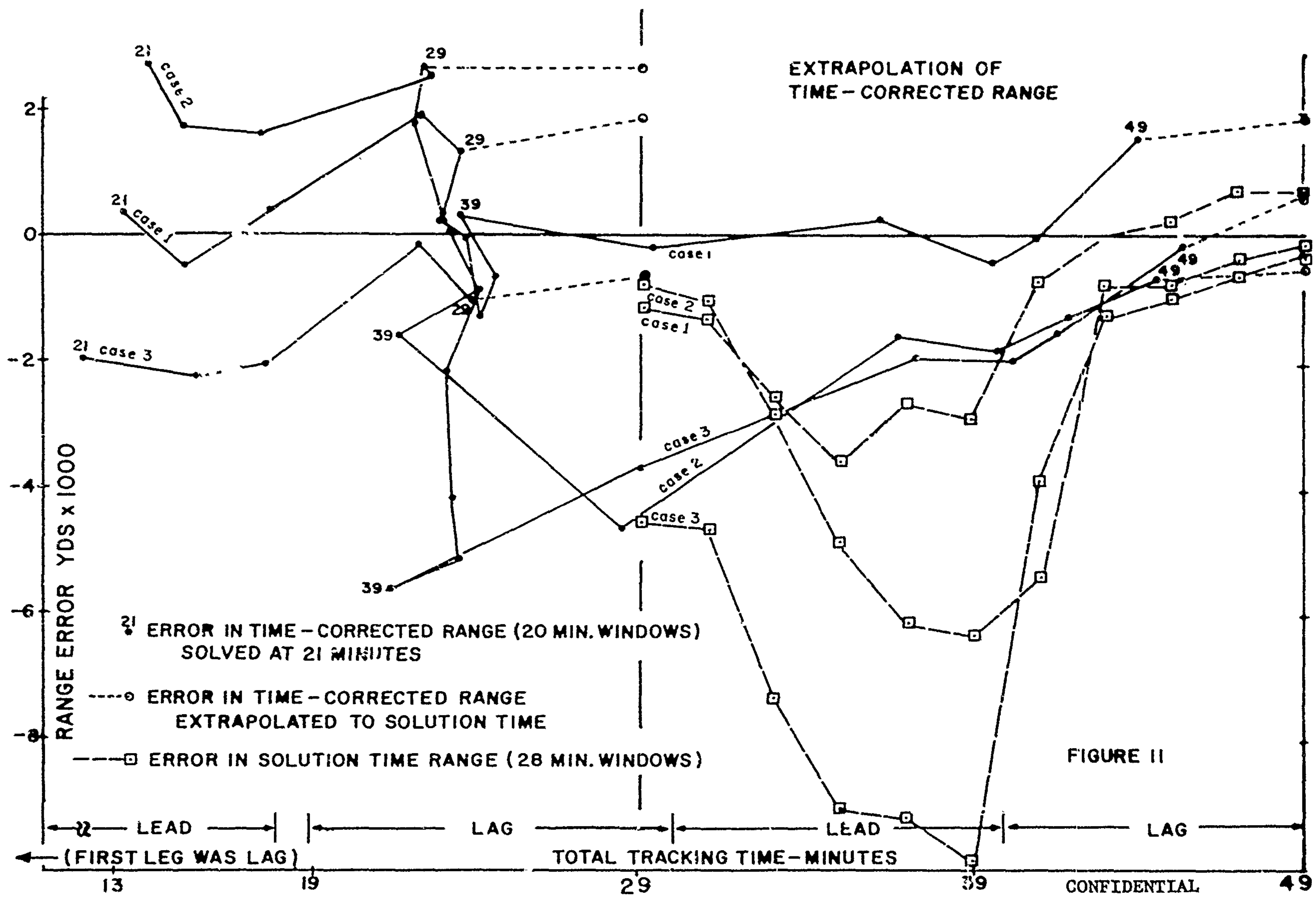


FIGURE II

CONFIDENTIAL

APPENDIX A

Quadruple-Bearing Range Solution

The following is a computing algorithm for obtaining range estimates from sets of four observed bearings, and for averaging with appropriate weights the estimates spanning a data window. The result is a range solution at best range time, i.e. a time-corrected range. While different in form, the equations used here are equivalent to those presented by D. C. Bossard in reference 2.

In a time interval (window) containing LW discrete equally-spaced observations, the observation times are designated

t_b = time at beginning of window

$t_{b+1} = t_b + \Delta t$

$t_i = t_{b+1} + i\Delta t$

Similar subscripts on other parameters refer to corresponding observation times. Further definitions are

B_i = bearing

x_{oi}, y_{oi} = own ship coordinates

$S_{ij} = \sin B_i \cos B_j - \cos B_i \sin B_j$

$C_{ij} = \cos B_i \cos B_j + \sin B_i \sin B_j$

$D_{ij} = (x_{oi}-x_{oj})\cos B_i - (y_{oi}-y_{oj})\sin B_i$

$L_{ij} = -(x_{oi}-x_{oj})\sin B_i - (y_{oi}-y_{oj})\cos B_i$

Using the above definitions compute the following for $1 \leqslant i \leqslant (LW-2)$:

$$D_i = S_{i,i-1} - C_{i-1,b}\, C_{i,b+1}\, S_{b+1,b}$$

$$\alpha_i = [L_{b,i-1}\, S_{i,i-1} - (D_{i,i-1} - D_{b+1,b}\, C_{i,b+1})C_{i-1,b}\,]/D_i$$

$$\beta_i = (t_{i-1} - t_b)\, S_{i,i-1}/D_i$$

$$\gamma_i = \Delta t\, C_{i-1,b}\, S_{i,b+1}/D_i$$

The range estimate for a single set of four bearings, although not explicitly used in the succeeding computation, is $(\hat{R}_b \cos \epsilon_b)$ in the equation

$$(\hat{R}_b \cos \epsilon_b)_i = \alpha_i + \beta_i (S_T I)_b + \gamma_i (S_T I)_{b+1}$$

$$+ (-C_{i-1,b}C_{i,b+1}C_{b+1,b};\ C_{i-1,b}C_{i,b-1};\ C_{i,b};\ -C_{i-1,b}) \begin{bmatrix} E_b \\ E_{b+1} \\ E_{i-1} \\ E_i \end{bmatrix} \div D_i$$

in which $S_T I$ is the line of sight component of target speed, and ϵ_b is the angle subtended by the cross range residual E_b.

We proceed next with the computation of weight. It is shown in reference (2) that the in-line range residual is approximately proportional to

$$\delta R_{bi} \approx (-\dot{E}_b + \dot{E}_{i-1})/D_i$$

This, together with the assumption that the covariance of $\dot{E}$ is

$$C_{ij}^{\dot{E}} \approx e^{-|t_i - t_j|/t_m} \triangleq \rho^{|i-j|}$$

leads to a covariance for the in-line range residual

$$C_{i,j}^{R} = (1 - \rho^{|i+1|} - \rho^{|j+1|} + \rho^{|i-j|})/(D_i\ D_j)$$

This is a symmetrical matrix in which

$$1 \leqslant i,j \leqslant (LW-2)$$

Compute the matrix and its inverse. Compute weights using the formula

$$w_i = \sum_{j=1}^{LW-2} (C^R)^{-1}_{ij} \Bigg/ \sum_{i=1}^{LW-2} \sum_{j=1}^{LW-2} (C^R)^{-1}_{ij}$$

Compute the weighted means

$$\overline{\alpha} = \sum_{i=1}^{LW-2} w_i \alpha_i$$

$$\overline{\beta} = \sum_{i=1}^{LW-2} w_i \beta_i$$

$$\overline{\gamma} = \sum_{i=1}^{LW-2} w_i \gamma_i$$

$$\overline{t} = t_b + \overline{\beta}$$

Interpolate for $\overline{B} = B(\overline{t})$. Then compute

$$t^* = \overline{t} + \overline{\gamma}\cos(\overline{B}-B_{b+1})/\cos(\overline{B}-B_b)$$

Interpolate for $B^* = B(t^*)$, and finally compute

$$R^* = \left[L_{b+1,\overline{t}} - L_{b+1,t^*}\right.$$

$$\left. + (\overline{\alpha} - L_{b\overline{t}})\cos(\overline{B} - B_{b+1})/\cos(\overline{B}-B_b)\right]/\cos(B^*-B_{b+1})$$

which is the time-corrected range for the window.

APPENDIX B

Justification of Exponential Correlation for $\dot{E}$

Consider a target vessel which is traveling at constant speed, making good a mean course while performing zigs as evasive maneuvers. The zig strategy is assumed to be as follows: the time for each zig is selected so that the probability of no zig in the time interval t_i to t_j is

$$P(\text{no zig between } t_i \text{ and } t_j) = e^{-|t_i - t_j|/t_m} \tag{1}$$

and the deviation from mean course resulting from each zig is drawn randomly from a rectangular distribution of variance $\sigma_{CT}^2 = \frac{1}{3}(\delta C_{T\,max})^2$. Then the covariance of target course deviation δC_T is

$$C_{ij}^{CT} = \epsilon[\delta C_T(t_i)\, \delta C_T(t_j)] \tag{2}$$

$$= \sigma_{CT}^2 P(\text{no zig between } t_i \text{ and } t_j)$$

$$+ (\text{zero}) P(\text{any zig between } t_i \text{ and } t_j) \tag{3}$$

$$= \sigma_{CT}^2 e^{-|t_i - t_j|/t_m} \tag{4}$$

The constant t_m turns out to be the mean time between zigs.

As a linearized approximation, the cross-range error residual E resulting from such target maneuvers may be written

$$E_i = \int_0^{t_i} S_{TI}\, \delta C_T dt + E_o \tag{5}$$

where S_{TI} is the component of target speed parallel to the bearing line. Then the time derivative of E is

$$\dot{E}_i = S_{TI}(t_i)\ \delta C_T(t_i) \qquad (6)$$

and the covariance of $\dot{E}$ is

$$C_{ij}^{\dot{E}} = \epsilon[\dot{E}_i \dot{E}_j] \qquad (7)$$

$$= \epsilon[S_{TI}(t_i)\ S_{TI}(t_j)\ \delta C_T(t_i)\ \delta(Ct_j)] \qquad (8)$$

If the approximation S_{TI} = constant is accepted, (8) becomes

$$C_{ij}^{\dot{E}} = S_{TI}^2\ \epsilon[\delta C_T(t_i)\ \delta C_T(t_j)] = S_{TI}^2\ C_{ij}^{CT} \qquad (9)$$

Substituting from (4),

$$C_{ij}^{\dot{E}} = S_{TI}^2\ \sigma_{CT}^2\ e^{-|t_i - t_j|/t_m} \qquad (10)$$

showing that the covariance of $\dot{E}$ arising from target zigs of the type described is approximately an exponential function of time difference.

APPENDIX C

Covariance of E Derived from Covariance of $\dot{E}$

Given that the covariance of the time derivative $\dot{E}$ of the error residual E is

$$C_{ij}^{\dot{E}} = \sigma_{\dot{E}}^{2} \, e^{-|t_i - t_j|/t_m} \qquad (1)$$

then the covariance of the residual itself is

$$C_{ij}^{E} = \epsilon(E_i E_j) = \epsilon\left[(E_o + \int_o^{t_i} \dot{E}dt)(E_o + \int_o^{t_j} \dot{E}dt)\right] \qquad (2)$$

$$= \epsilon\left[E_o^2 + E_o(E_i + E_j - 2E_o)\right] + \int_o^{t_i}\int_o^{t_j} \epsilon\left[\dot{E}(t)\dot{E}(t')\right] dtdt' \qquad (3)$$

$$= C_{oi}^{E} + C_{oj}^{E} - C_{oo}^{E} + \sigma_{\dot{E}}^{2} \int_o^{t_j}\left[\int_o^{t_i} e^{-|t'-t|/t_m} dt\right] dt' \qquad (4)$$

Integration will be performed piecewise over regions within which the sign of the exponent can be insured. The double integral becomes, if $t_i \geqslant t_j \geqslant 0$

$$\int_o^{t_j}\left[\int_o^{t'} e^{(t-t')/t_m} dt + \int_{t'}^{t_i} e^{-(t-t')/t_m} dt\right] dt'$$

$$= t_m \int_o^{t_j}\left[e^{-t'/t_m} - e^{-(t_i - t')/t_m} + 2\right] dt'$$

$$= t_m^2 \left(-e^{-(t_i - t_j)/t_m} + e^{-t_i/t_m} + e^{-t_j/t_m} - 1\right) + 2t_m t_j \qquad (5)$$

If $t_j \geqslant t_i \geqslant 0$, symmetry with respect to i and j leads to the analog of (5)

$$t_m^2 (-e^{-(t_j-t_i)/t_m} + e^{-t_j/t_m} - 1) + 2t_m t_i \tag{6}$$

If $0 \geqslant t_j \geqslant t_i$, the integral of (4) becomes

$$\int_{t_j}^{o} \left[\int_{t_i}^{t'} e^{(t-t')/t_m} dt + \int_{t'}^{o} e^{-(t-t')/t_m} dt \right] dt'$$

$$= t_m \int_{t_j}^{o} \left[-e^{(t_i-t')/t_m} - e^{t'/t_m} + 2 \right] dt'$$

$$= t_m^2 (-e^{(t_i-t_j)/t_m} + e^{t_i/t_m} + e^{t_j/t_m} - 1) - 2t_m t_j \tag{7}$$

By symmetry, if $0 \geqslant t_i \geqslant t_j$ the analog of (7) is

$$t_m^2 (-e^{(t_j-t_i)/t_m} + e^{t_i/t_m} + e^{t_j/t_m} - 1) - 2t_m t_i \tag{8}$$

If $t_i \geqslant 0 \geqslant t_j$, the integral of (4) becomes

$$\int_{o}^{t_j} \left[\int_{o}^{t_i} e^{-(t-t')/t_m} dt \right] dt'$$

$$= t_m \int_{o}^{t_j} \left[-e^{-(t_i-t')/t_m} + e^{t'/t_m} \right] dt'$$

$$= t_m^2 (-e^{-(t_j-t_i)/t_m} + e^{t_i/t_m} + e^{-t_j/t_m} - 1) \tag{9}$$

Again by symmetry, if $t_j \geqslant 0 \geqslant t_i$ we obtain the analog of (9):

$$t_m^2 (-e^{-(t_j-t_i)/t_m} + e^{t_i/t_m} + e^{-t_j/t_m} - 1) \qquad (10)$$

Expressions (5) through (10) can all be represented by

$$t_m^2 (-e^{-|t_i-t_j|/t_m} + e^{|t_i|/t_m} + e^{|t_j|/t_m} - 1) + t_m\theta \qquad (11)$$

where $t_m\theta$ represents the term following the parenthesis in (5) through (8), and is zero in cases (9) and (10). Empirically one may verify that

$$\theta = |t_i| + |t_j| - |t_i-t_j| \qquad (12)$$

fits the requirements. Substituting (11) and (12) into (4),

$$C_{ij}^E = C_{io}^E + C_{oj}^E - C_{oo}^E$$

$$+ \sigma_E^2 t_m^2 \Big[-e^{-|t_i-t_j|/t_m} + e^{-|t_i|/t_m} + e^{-|t_j|/t_m} - 1$$

$$+ (|t_i| + |t_j| - |t_i-t_j|)/t_m \Big] \qquad (13)$$

APPENDIX D

Covariance of E for $\overline{E} = 0$ Over Selected Period

In a previous derivation the expression

$$C^E_{ij} = C^E_{io} + C^E_{oj} - C^E_{oo}$$

$$+ \sigma_{\dot{E}}^2 \, t_m^2 \Big[-e^{-|t_i - t_j|/t_m} + e^{-|t_i|/t_m} + e^{-|t_j|/t_m} - 1$$

$$+ (|t_i| + |t_j| - |t_i - t_j|)/t_m \Big] \tag{1}$$

was obtained for the covariance of E, corresponding to a given covariance for the time derivative of E:

$$C^{\dot{E}}_{ij} = \sigma_{\dot{E}}^2 \, e^{-|t_i - t_j|/t_m} \tag{2}$$

C^E_{io} stands for a function of t_i only, C^E_{oj} a function of t_j only, and C^E_{oo} a constant. These arbitrary quantities are not further restricted by equation (2), nor by the general requirements for autocovariance functions except that C^E_{ij} must be positive definite. In order to get explicit expressions for the arbitrary quantities, another requirement which seems reasonable is here imposed: that the mean value of E over a selected interval $t_a - \tau$ to $t_a + \tau$ is zero,

$$\overline{E} = \frac{1}{2\tau} \int_{t_a - \tau}^{t_a + \tau} E_i dt_i = 0 \tag{3}$$

The covariance which meets this requirement will be designated $\widetilde{C}^E$. From (3) it follows that

$$\epsilon(E_j E) = \frac{1}{2\tau}\int_{t_a-\tau}^{t_a-\tau}\epsilon(E_i E_j)dt_i = 0 \tag{4}$$

$$0 = \frac{1}{2\tau}\int_{t_a-\tau}^{t_a+\tau} C_{ij}dt_i \tag{5}$$

Before substituting (1) into (5), equation (1) is abbreviated by replacing with F_i, F_j, and K all of the terms in t_i only, terms in t_j only, and constant terms, respectively. Equation (1) becomes

$$\widetilde{C}^E_{ij} = \sigma_{\dot{E}}^2\ t_m^2\left[-e^{-|t_i-t_j|/t_m} - |t_i-t_j|/t_m + F_i + F_j + K\right] \tag{6}$$

which is substituted into 5, dropping constant factors:

$$0 = \int_{t_a-\tau}^{t_a+\tau}\left[-e^{-|t_i-t_j|/t_m} \cdot |t_i-t_j|/t_m + F_i + F_j + K\right]dt_i \tag{7}$$

$$= \int_{t_a-\tau}^{t_j}\left[-e^{(t_i-t_j)/t_m} + (t_i-t_j)/t_m\right]dt_i$$

$$+ \int_{t_j}^{t_a+\tau}\left[-e^{-(t_i-t_j)/t_m} - (t_i-t_j)/t_m\right]dt_i$$

$$+ \int_{t_a-\tau}^{t_a+\tau} F_i dt_j + 2\tau F_j + 2\tau K \tag{8}$$

$$= \left[-t_m e^{(t_i - t_j)/t_m} + t_i^2/2t_m - t_i t_j/t_m \right]_{t_a - \tau}^{t_j}$$

$$+ \left[t_m e^{(t_i - t_j)/t_m} - t_i^2/2t_m + t_i t_j/t_m \right]_{t_j}^{t_a + \tau} \tag{9}$$

$$+ \int_{t_a - \tau}^{t_a + \tau} F_i dt_i + 2\tau F_j + 2\tau K$$

$$= -t_m \left[1 - e^{(t_a - \tau - t_j)/t_m} \right] + \left[t_j^2 - (t_a - \tau)^2 \right] \Big/ 2t_m - \left[t_j^2 - t_j(t_a - \tau) \right] \Big/ t_m$$

$$+ t_m \left[e^{-(t_a + \tau - t_j)/t_m} \right] - \left[(t_a + \tau)^2 - t_j^2 \right] \Big/ 2t_m + \left[t_j(t_a + \tau) - t_j^2 \right] \Big/ t_m$$

$$+ \int_{t_a - \tau}^{t_a + \tau} F_i dt_i + 2\tau F_j + 2\tau K \tag{10}$$

$$0 = t_m \left\{ e^{-\tau/t_m} \left[e^{(t_j - t_a)/t_m} + e^{-(t_j - t_a)/t_m} \right] - 2 \right\}$$

$$+ (t_j - t_a)^2/t_m - \tau^2/t_m \tag{11}$$

$$+ \int_{t_a - \tau}^{t_a + \tau} F_i dt_i + 2\tau F_j + 2\tau K$$

By symmetry, (11) may also be written with i and j interchanged:

$$0 = t_m \left\{ e^{-\tau/t_m} \left[e^{(t_i - t_a) t_m} + e^{-(t_i - t_a)/t_m} \right] - 2 \right\}$$

$$- (t_i - t_a)^2/t_m - \tau^2/t_m \qquad (12)$$

$$+ \int_{t_a - \tau}^{t_a + \tau} F_j dt_j + 2\tau F_i + 2\tau K$$

Subtracting (11) from (12), noting that $\int F_i dt_i = \int F_j dt_j$:

$$0 = t_m e^{-\tau/t_m} \left[e^{(t_i - t_a)/t_m} + e^{-(t_i - t_a)/t_m} - e^{(t_j - t_a)/t_m} - e^{-(t_j - t_a)/t_m} \right]$$

$$-(t_i - t_a)^2/t_m + (t_j - t_a)^2/t_m + 2\tau \left[F_i - F_j \right] \qquad (13)$$

Because t_i and t_j are independent, the terms in t_i must sum to zero or a constant, and those in t_j also must sum to zero or to an offsetting constant. We assume that the constant part, if any, is assigned to K. Then

$$e^{-\tau/t_m} \left[e^{(t_i - t_a)/t_m} + e^{-(t_i - t_a)/t_m} \right] - (t_i - t_a)^2/t_m + 2\tau F_i = 0 \qquad (14)$$

$$F_i = -(t_m/2\tau) e^{-\tau/t_m} \left[e^{(t_i - t_a)/t_m} + e^{-(t_i - t_a)/t_m} \right] + (t_i - t_a)/2 t_m \tau \qquad (15)$$

An analogous expression applies for F_j.

To evaluate K, substitute (15) and its j-analog into (11):

$$0 = t_m\left\{e^{-\tau/t_m}\left[e^{(t_j-t_a)/t_m} + e^{-(t_j-t_a)/t_m}\right] - 2\right\} - (t_j-t_a)^2/t_m - \tau^2/t_m$$

$$+ \int_{t_a-\tau}^{t_a+\tau}\left\{-(t_m/2\tau)e^{-\tau/t_m}\left[e^{(t_i-t_a)/t_m} + e^{-(t_i-t_a)/t_m}\right]\right.$$

$$\left. + (t_i-t_a)^2/2t_m\tau\right\}dt_i + 2\tau\left\{-(t_m/2\tau)e^{-\tau/t_m}\left[e^{(t_j-t_a)/t_m} + e^{-(t_j-t_a)/t_m}\right]\right.$$

$$\left. + (t_j-t_a)^2/2t_m\tau\right\} + 2\tau K \qquad (16)$$

$$0 = -2t_m - \tau^2/t_m - (t_m^2/2\tau)e^{-\tau/t_m}\left[-e^{(t_i-t_a)/t_m} + e^{-(t_i-t_a)/t_m}\right]_{t_a-\tau}^{t_a+\tau}$$

$$+ (1/t_m\tau)\left[t_i^3/6 - t_a t_i^2/2 + t_a^2 t_i/2\right]_{t_a-\tau}^{t_a+\tau} + 2\tau K \qquad (17)$$

$$0 = -2t_m - \tau^2/t_m - (t_m^2/\tau)\quad e^{-\tau/t_m}\left[e^{\tau/t_m} - e^{-\tau/t_m}\right]$$

$$+ (1/6t_m\tau)\left[(t_a+\tau)^3 - (t_a-\tau)^3\right] - (t_a/2t_m\tau)\left[(t_a+\tau)^2 - (t_a-\tau)^2\right]$$

$$+ (t_a^2/2t_m\tau)\left[(t_a+\tau) - (t_a-\tau)\right] + 2\tau K \qquad (18)$$

$$K = t_m/\tau + \tau/3t_m + (t_m^2/2\tau^2)\left[1-e^{-2\tau/t_m}\right] \qquad (19)$$

If equation (15) and its j-analog and (19) are substituted into (6), the complete expression for covariance is obtained:

$$\tilde{C}_{ij}^{E} = \sigma_{E}^{2} t_m^{2} \Big\{ -e^{-|t_i-t_j|/t_m} - |t_i-t_j|/t_m$$

$$- (t_m/2\tau)e^{-\tau/t_m}\left[e^{(t_i-t_a)/t_m} + e^{-(t_i-t_a)/t_m}\right] + (t_i-t_a)^2/2t_m\tau$$

$$- (t_m/2\tau)e^{-\tau/t_m}\left[e^{(t_j-t_a)/t_m} + e^{-(t_j-t_a)/t_m}\right] + (t_j-t_a)^2/2t_m\tau$$

$$+ t_m/\tau + \tau/3t_m + (t_m^2/2\tau^2)\left[1 - e^{-2\tau/t_m}\right]\Big\} \qquad (20)$$

If t_a is taken as the time origin and the window of length w coincides with the interval $t_a-\tau$ to $t_a+\tau$, then (20) simplifies to

$$\tilde{C}_{ij}^{E} = \sigma_{E}^{2} t_m^{2} \Big\{ -e^{-|t_i-t_j|/t_m} - |t_i-t_j|/t_m$$

$$- (t_m/w)\, e^{-w/2t_m}\left[e^{t_i} + e^{-t_i} + e^{t_j} + e^{-t_j}\right] + (t_i^2 + t_j^2)/t_m w$$

$$+ 2t_m/w + w/6t_m + (2t_m^2/w^2)\left[1 - e^{-w/t_m}\right]\Big\} \qquad (21)$$

Equation (21) has been used for some of the reported tests.

CE	.1361E+01	.1292E+01	.1218E+01	.1143E+01	.1069E+01	.1000E+01	.9391E+00	.8858E+00	.8392E+00	.7984E+00	.7627E+00
	.1292E+01	.1241E+01	.1182E+01	.1120E+01	.1058E+01	.1000E+01	.9483E+00	.9030E+00	.8634E+00	.8288E+00	.7984E+00
	.1218E+01	.1182E+01	.1141E+01	.1095E+01	.1046E+01	.1000E+01	.9587E+00	.9226E+00	.8910E+00	.8634E+00	.8392E+00
	.1143E+01	.1120E+01	.1095E+01	.1065E+01	.1033E+01	.1000E+01	.9707E+00	.9451E+00	.9226E+00	.9030E+00	.8858E+00
	.1069E+01	.1058E+01	.1046E+01	.1033E+01	.1017E+01	.1000E+01	.9844E+00	.9707E+00	.9587E+00	.9483E+00	.9391E+00
	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
	.9391E+00	.9483E+00	.9587E+00	.9707E+00	.9844E+00	.1000E+01	.1017E+01	.1033E+01	.1046E+01	.1058E+01	.1069E+01
	.8858E+00	.9030E+00	.9226E+00	.9451E+00	.9707E+00	.1000E+01	.1033E+01	.1065E+01	.1095E+01	.1120E+01	.1143E+01
	.8392E+00	.8634E+00	.8910E+00	.9226E+00	.9587E+00	.1000E+01	.1046E+01	.1095E+01	.1141E+01	.1182E+01	.1218E+01
	.7984E+00	.8288E+00	.8634E+00	.9030E+00	.9483E+00	.1000E+01	.1058E+01	.1120E+01	.1182E+01	.1241E+01	.1292E+01
	.7627E+00	.7984E+00	.8392E+00	.8858E+00	.9391E+00	.1000E+01	.1069E+01	.1143E+01	.1218E+01	.1292E+01	.1361E+01

.3851E+03	-.8252E+03	.5578E+03	-.1493E+03	.3997E+02	-.1070E+02	.2864E+01	-.7663E+00	.2042E+00	-.5128E-01	.8723E-02
-.8252E+03	.2153E+04	-.2020E+04	.8778E+03	-.2350E+03	.6289E+02	-.1683E+02	.4505E+01	-.1200E+01	.3014E+00	-.5128E-01
.5578E+03	-.2020E+04	.2961E+04	-.2237E+04	.9357E+03	-.2505E+03	.6704E+02	-.1794E+02	.4781E+01	-.1200E+01	.2042E+00
-.1493E+03	.8778E+03	-.2237E+04	.3019E+04	-.2252E+04	.9399E+03	-.2516E+03	.6732E+02	-.1794E+02	.4505E+01	-.7663E+00
.3997E+02	-.2350E+03	.9357E+03	-.2252E+04	.3023E+04	-.2253E+04	.9401E+03	-.2516E+03	.6704E+02	-.1683E+02	.2864E+01
-.1070E+02	.6289E+02	-.2505E+03	.9398E+03	-.2253E+04	.3025E+04	-.2253E+04	.9398E+03	-.2505E+03	.6289E+02	-.1070E+02
.2864E+01	-.1683E+02	.6704E+02	-.2516E+03	.9401E+03	-.2253E+04	.3023E+04	-.2252E+04	.9357E+03	-.2350E+03	.3997E+02
-.7663E+00	.4505E+01	-.1794E+02	.6732E+02	-.2516E+03	.9399E+03	-.2252E+04	.3019E+04	-.2237E+04	.8778E+03	-.1493E+03
.2042E+00	-.1200E+01	.4781E+01	-.1794E+02	.6704E+02	-.2505E+03	.9357E+03	-.2237E+04	.2961E+04	-.2020E+04	.5578E+03
-.5128E-01	.3014E+00	-.1200E+01	.4505E+01	-.1683E+02	.6289E+02	-.2350E+03	.8778E+03	-.2020E+04	.2153E+04	-.8252E+03
.8723E-02	-.5128E-01	.2042E+00	-.7663E+00	.2864E+01	-.1070E+02	.3997E+02	-.1493E+03	.5578E+03	-.8252E+03	.3851E+03

Figure 1 - C^{E}_{ij} Simple and its inverse

$t_m = 15$

$t_i = 2i, \qquad -5 \leq i,j \leq 5$

CE	.3228E-00	.2595E-00	.1916E-00	.1220E-00	.5280E-01	-.1395E-01	-.7688E-01	-.1349E-00	-.1875E-00	-.2344E-00	-.2756E-00
	.2595E-00	.2133E-00	.1611E-00	.1051E-00	.4788E-01	-.8405E-02	-.6217E-01	-.1122E-00	-.1578E-00	-.1985E-00	-.2344E-00
	.1916E-00	.1611E-00	.1259E-00	.8550E-01	.4198E-01	-.2342E-02	-.4564E-01	-.8654E-01	-.1241E-00	-.1578E-00	-.1875E-00
	.1220E-00	.1051E-00	.8550E-01	.6217E-01	.3428E-01	.3631E-02	-.2770E-01	-.5813E-01	-.8654E-01	-.1122E-00	-.1349E-00
	.5280E-01	.4788E-01	.4198E-01	.3428E-01	.2346E-01	.8428E-02	-.9234E-02	-.2770E-01	-.4564E-01	-.6217E-01	-.7688E-01
	-.1395E-01	-.8405E-02	-.2342E-02	.3631E-02	.8428E-02	.1047E-01	.8428E-02	.3631E-02	-.2342E-02	-.8405E-02	-.1395E-01
	-.7688E-01	-.6217E-01	-.4564E-01	-.2770E-01	-.9234E-02	.8428E-02	.2346E-01	.3428E-01	.4198E-01	.4788E-01	.5280E-01
	-.1349E-00	-.1122E-00	-.8654E-01	-.5813E-01	-.2770E-01	.3631E-02	.3428E-01	.6217E-01	.8550E-01	.1051E-00	.1220E-00
	-.1875E-00	-.1578E-00	-.1241E-00	-.8654E-01	-.4564E-01	-.2342E-02	.4198E-01	.8550E-01	.1259E-00	.1611E-00	.1916E-00
	-.2344E-00	-.1985E-00	-.1578E-00	-.1122E-00	-.6217E-01	-.8405E-02	.4788E-01	.1051E-00	.1611E-00	.2133E-00	.2595E-00
	-.2756E-00	-.2344E-00	-.1875E-00	-.1349E-00	-.7688E-01	-.1395E-01	.5280E-01	.1220E-00	.1916E-00	.2595E-00	.3228E-00

W	.2263E+04	.4498E+04	.5111E+04	.4610E+04	.4743E+04	.4711E+04	.4706E+04	.4759E+04	.4553E+04	.5323E+04	.1878E+04
	.4498E+04	.1724E+05	.1088E+05	.1437E+05	.1309E+05	.1345E+05	.1331E+05	.1349E+05	.1290E+05	.1509E+05	.5323E+04
	.5111E+04	.1088E+05	.1400E+05	.9300E+04	.1234E+05	.1120E+05	.1147E+05	.1152E+05	.1104E+05	.1290E+05	.4553E+04
	.4610E+04	.1437E+05	.9300E+04	.1508E+05	.9665E+04	.1290E+05	.1167E+05	.1213E+05	.1152E+05	.1349E+05	.4759E+04
	.4743E+04	.1309E+05	.1234E+05	.9665E+04	.1480E+05	.9570E+04	.1272E+05	.1167E+05	.1147E+05	.1331E+05	.4706E+04
	.4711E+04	.1345E+05	.1120E+05	.1290E+05	.9570E+04	.1489E+05	.9570E+04	.1290E+05	.1120E+05	.1345E+05	.4711E+04
	.4706E+04	.1331E+05	.1147E+05	.1167E+05	.1272E+05	.9570E+04	.1480E+05	.9665E+04	.1234E+05	.1309E+05	.4743E+04
	.4759E+04	.1349E+05	.1152E+05	.1213E+05	.1167E+05	.1290E+05	.9665E+04	.1508E+05	.9300E+04	.1437E+05	.4610E+04
	.4553E+04	.1290E+05	.1104E+05	.1152E+05	.1147E+05	.1120E+05	.1234E+05	.9300E+04	.1400E+05	.1088E+05	.5111E+04
	.5323E+04	.1509E+05	.1290E+05	.1349E+05	.1331E+05	.1345E+05	.1309E+05	.1437E+05	.1098E+05	.1724E+05	.4498E+04
	.1878E+04	.5323E+04	.4553E+04	.4759E+04	.4706E+04	.4711E+04	.4743E+04	.4610E+04	.5111E+04	.4498E+04	.2263E+04

Figure 2 - $\tilde{C}^E_{ij}$ and its inverse

$t_m = 15$, $\dot{E}^2/t_m^2 = 1$, $t_a = 0$

$t_i = 2i$, $-5 \leqslant i,j \leqslant 5$

CE	.1363E+01	.1292E+01	.1218E+01	.1143E+01	.1069E+01	.1000E+01	.9391E+00	.8858E+00	.8392E+00	.7984E+00	.7627E+00
	.1292E+01	.1243E+01	.1182E+01	.1120E+01	.1058E+01	.1000E+01	.9483E+00	.9030E+00	.8634E+00	.8288E+00	.7984E+00
	.1218E+01	.1182E+01	.1143E+01	.1095E+01	.1046E+01	.1000E+01	.9587E+00	.9226E+00	.8910E+00	.8634E+00	.8392E+00
	.1143E+01	.1120E+01	.1095E+01	.1067E+01	.1033E+01	.1000E+01	.9707E+00	.9451E+00	.9226E+00	.9030E+00	.8858E+00
	.1069E+01	.1058E+01	.1046E+01	.1033E+01	.1019E+01	.1000E+01	.9844E+00	.9707E+00	.9587E+00	.9483E+00	.9391E+00
	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1002E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01	.1000E+01
	.9391E+00	.9483E+00	.9587E+00	.9707E+00	.9844E+00	.1000E+01	.1019E+01	.1033E+01	.1046E+01	.1058E+01	.1069E+01
	.8858E+00	.9030E+00	.9226E+00	.9451E+00	.9707E+00	.1000E+01	.1033E+01	.1067E+01	.1095E+01	.1120E+01	.1143E+01
	.8392E+00	.8634E+00	.8910E+00	.9226E+00	.9587E+00	.1000E+01	.1046E+01	.1095E+01	.1143E+01	.1182E+01	.1218E+01
	.7984E+00	.8288E+00	.8634E+00	.9030E+00	.9483E+00	.1000E+01	.1058E+01	.1120E+01	.1182E+01	.1243E+01	.1292E+01
	.7627E+00	.7984E+00	.8392E+00	.8858E+00	.9391E+00	.1000E+01	.1069E+01	.1143E+01	.1218E+01	.1292E+01	.1363E+01
W	.9832E+02	-.1321E+03	-.1097E+01	.2214E+02	.1196E+02	.2544E+01	-.7063E+00	-.8163E+00	-.3170E+00	-.2467E-01	.7876E-01
	-.1321E+03	.2758E+03	-.1306E+03	-.3085E+02	.6074E+01	.8532E+01	.3487E+01	.3806E+00	-.4076E+00	-.2826E+00	-.2467E-01
	-.1097E+01	-.1306E+03	.2758E+03	-.1309E+03	-.3099E+02	.6044E+01	.8544E+01	.3503E+01	.3909E+00	-.4076E+00	-.3170E+00
	.2214E+02	-.3085E+02	-.1309E+03	.2808E+03	-.1282E+03	-.3038E+02	.5915E+01	.8406E+01	.3503E+01	.3806E+00	-.8163E+00
	.1196E+02	.6074E+01	-.3099E+02	-.1282E+03	.2823E+03	-.1278E+03	-.3041E+02	.5915E+01	.8544E+01	.3487E+01	-.7063E+00
	.2544E+01	.8532E+01	.6044E+01	-.3038E+02	-.1278E+03	.2825E+03	-.1278E+03	-.3038E+02	.6044E+01	.8532E+01	.2544E+01
	-.7063E+00	.3487E+01	.8544E+01	.5915E+01	-.3041E+02	-.1278E+03	.2823E+03	-.1282E+03	-.3099E+02	.6074E+01	.1196E+02
	-.8163E+00	.3806E+00	.3503E+01	.8406E+01	.5915E+01	-.3038E+02	-.1282E+03	.2808E+03	-.1309E+03	-.3085E+02	.2214E+02
	-.3170E+00	-.4076E+00	.3909E+00	.3503E+01	.8544E+01	.6044E+01	-.3099E+02	-.1309E+03	.2758E+03	-.1306E+03	-.1097E+01
	-.2467E-01	-.2826E+00	-.4076E+00	.3806E+00	.3487E+01	.8532E+01	.6074E+01	-.3085E+02	-.1306E+03	.2758E+03	-.1321E+03
	.7876E-01	-.2467E-01	-.3170E+00	-.8163E+00	-.7063E+00	.2544E+01	.1196E+02	.2214E+02	-.1097E+01	-.1321E+03	.9832E+02

Figure 3 - (C^E_{ij}Simple + .002I) and its inverse

$t_m = 15$

$t_i = 2i, \qquad -5 \leq i,j \leq 5$

CE	.3248E+00	.2595E+00	.1916E+00	.1220E+00	.5280E-01	-.1395E-01	-.7688E-01	-.1349E+00	-.1875E+00	-.2344E+00	-.2756E+00
	.2595E+00	.2153E+00	.1611E+00	.1051E+00	.4788E-01	-.8405E-02	-.6217E-01	-.1122E+00	-.1578E+00	-.1985E+00	-.2344E+00
	.1916E+00	.1611E+00	.1279E+00	.8550E-01	.4198E-01	-.2342E-02	-.4564E-01	-.8654E-01	-.1241E+00	-.1578E+00	-.1875E+00
	.1220E+00	.1051E+00	.8550E-01	.6417E-01	.3428E-01	.3631E-02	-.2770E-01	-.5813E-01	-.8654E-01	-.1122E+00	-.1349E+00
	.5280E-01	.4788E-01	.4198E-01	.3428E-01	.2545E-01	.8428E-02	-.9234E-02	-.2770E-01	-.4564E-01	-.6217E-01	-.7688E-01
	-.1395E-01	-.8405E-02	-.2342E-02	.3631E-02	.8428E-02	.1247E-01	.8428E-02	.3631E-02	-.2342E-02	-.8405E-02	-.1395E-01
	-.7688E-01	-.6217E-01	-.4564E-01	-.2770E-01	-.9234E-02	.8428E-02	.2545E-01	.3428E-01	.4198E-01	.4788E-01	.5280E-01
	-.1349E+00	-.1122E+00	-.8654E-01	-.5813E-01	-.2770E-01	.3631E-02	.3428E-01	.6417E-01	.8550E-01	.1051E+00	.1220E+00
	-.1875E+00	-.1578E+00	-.1241E+00	-.8654E-01	-.4564E-01	-.2342E-02	.4198E-01	.8550E-01	.1279E+00	.1611E+00	.1916E+00
	-.2344E+00	-.1985E+00	-.1578E+00	-.1122E+00	-.6217E-01	-.8405E-02	.4788E-01	.1051E+00	.1611E+00	.2153E+00	.2595E+00
	-.2756E+00	-.2344E+00	-.1875E+00	-.1349E+00	-.7688E-01	-.1395E-01	.5280E-01	.1220E+00	.1916E+00	.2595E+00	.3248E+00

W	.1142E+03	-.1064E+03	.2825E+02	.5175E+02	.4107E+02	.3143E+02	.2841E+02	.2879E+02	.2903E+02	.2562E-02	.1593E+02
	-.1064E+03	.3173E+03	-.8314E+02	.1706E+02	.5318E+02	.5527E+02	.5060E+02	.4829E+02	.4707E+02	.4121E+02	.2562E+02
	.2825E+02	-.8314E+02	.3301E+03	-.7605E+02	.2291E+02	.5952E+02	.6244E+02	.5832E+02	.5472E+02	.4707E+02	.2903E+02
	.5175E+02	.1706E+02	-.7605E+02	.3361E+03	-.7379E+02	.2355E+02	.6028E+02	.6371E+02	.5832E+02	.4829E+02	.2879E+02
	.4107E+02	.5318E+02	.2291E+02	-.7379E+02	.3357E+03	-.7482E+02	.2300E+02	.6028E+02	.6244E+02	.5060E+02	.2841E+02
	.3143E+02	.5527E+02	.5952E+02	.2355E+02	-.7482E+02	.3349E+03	-.7482E+02	.2355E+02	.5952E+02	.5527E+02	.3143E+02
	.2841E+02	.5060E+02	.6244E+02	.6028E+02	.2300E+02	-.7482E+02	.3357E+03	-.7379E+02	.2291E+02	.5318E+02	.4107E+02
	.2879E+02	.4829E+02	.5832E+02	.6371E+02	.6028E+02	.2355E+02	-.7379E+02	.3361E+03	-.7605E+02	.1706E+02	.5175E+02
	.2903E+02	.4707E+02	.5472E+02	.5832E+02	.6244E+02	.5952E+02	.2291E+02	-.7605E+02	.3301E+03	-.8314E+02	.2825E+02
	.2562E+02	.4121E+02	.4707E+02	.4829E+02	.5060E+02	.5527E+02	.5318E+02	.1706E+02	-.8314E+02	.3173E+03	-.1064E+03
	.1593E+02	.2562E+02	.2903E+02	.2879E+02	.2841E+02	.3143E+02	.4107E+02	.5175E+02	.2825E+02	-.1064E+03	.1142E+03

Figure 4 - $(\tilde{C}^E_{ij} + .002I)$ and its inverse

$t_m = 15$, $\dot{E}^2/t_m^2 = 1$, $t_a = 0$

$t_i = 2i$ $-5 \leqslant i,j \leqslant 5$

APPENDIX F

Glossary of Symbols

a_1, a_3 - east and north components, respectively, of target velocity.

a_2, a_4 - east and north components, respectively, of target position with respect to an arbitrary fixed origin.

a (matrix notation) - $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$

A (matrix notation) - an n-row matrix having rows of the form $t_i \cos B_i$, $\cos B_i$, $-t_i \sin B_i$, $-\sin B_i$.

B_i - observed bearing to target at time t_i.

C (Appendix A) - an abbreviation for several terms, defined as introduced.

C^E, $C^{\dot{E}}$, etc. - covariance of the superscribed variable.

C^E Simple - a particular formula for the covariance of E, derived in Appendix C.

$\tilde{C}^E$ - a particular formula for the covariance of E, derived in Appendix B.

C_T - target course

D (Appendix A)	-	an abbreviation for several terms, defined as introduced.
d (matrix notation)	-	an n-element column vector with elements of form $x_{oi} \cos B_i - y_{oi} \sin B_{oi}$.
E	-	the cross range miss distance of the estimated bearing line. In matrix notation, a column vector with elements E_i.
g (matrix notation)	-	the constant term in the matrix representation of CHURN normal equations.
I (matrix notation)	-	the identity matrix, having all ones on the principal diagonal, all zeros elsewhere.
L (Appendix A)	-	an abbreviation for several terms, defined as introduced.
P (Appendix A)	-	an abbreviation for several terms, defined as introduced.
R	-	range to target.
R*	-	estimated range at best range time, e.g. time-corrected range.
$R(\tau)$	-	autocorrelation function.
$R(\tau)_{sample}$	-	estimate of $R(\tau)$ based on a particular sample.
S_{TI}	-	component of target speed along bearing line.
S (Appendix A)	-	an abbreviation for several terms, defined as introduced.

t	-	time.
t_m	-	mean time interval between zigs.
t^*	-	best range time: time at which the expected range error of a solution is least.
W (matrix notation)	-	a weighting matrix, usually the reciprocal of the assumed covariance of residuals.
x_o, y_o	-	coordinates of tracking ship with respect to an arbitrary origin.
α	-	a combining factor used when adding covariance contributions from target zig and from bearing noise.
α, β, γ (Appendix A)	-	an abbreviation for several terms, defined as introduced.
δ	-	(used as prefix) indicates error quantity.
ϵ	-	expected value operator.
ϵ_b (Appendix A)	-	angle subtended by the cross range residual E_b.
θ (Appendix C)	-	symbol for an unknown expression, dropped when the expression is determined.
ρ	-	e^{-2/t_m}.
σ_E^2, $\sigma_{\dot{E}}^2$, etc.	-	variance of the parameter indicated by subscript.
τ	-	time interval.
ψ (matrix notation)	-	4 x 4 matrix, coefficient of the vector a in the CHURN normal equations.

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13. ABSTRACT

research

(C) In a new sonar bearings-only solution method, Dr. ~~D. G. Bossard of Daniel Wagner Associates~~ achieved ~~quite spectacular~~ reduction in range errors on a zigging target, one-sixth those of the usual (unweighted) CHURN method. ~~His~~ method yields time-corrected range (value at time when expected error is least) and weights observations according to assumed zig statistics. ~~Bossard~~ also advocates extrapolating favorably-chosen time-corrected ranges to obtain present range.

It was found those used in the research

(~~C~~) ~~We find~~ that the CHURN, with weights equivalent to ~~Bossard's~~, achieves equally small time-corrected range errors, and errors at solution time one-third those of the usual CHURN. Random bearing noise, however, seriously degrades solutions using ~~Bossard~~ weights, even without zigs, in which case the unweighted CHURN is optimum. For combinations of zigs and bearing noise, optimum combined weighting functions exist.

(~~C~~) Unsuccessful attempts were made to use data available to the tracking ship, e.g. autocorrelation of solution residuals, for selecting optimum weighting. Autocorrelation was also probed for zig detection clues without success.

(~~C~~) Results obtained by extrapolating pairs of time-corrected range to present time were about equally as good as from single solutions using the same data.

(~~C~~) It was ~~We~~ concluded that the ~~Bossard's~~ important contribution of the research is to show the effectiveness of appropriate statistical weighting.

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